Math 131 Final

Thursday, March 20

Time: 3 hours
Total: 80 points

This is a closed book test. You may use without proof any thing done in class, or in sections of the book included for this exam. Calculators, cell phones etc. are not allowed. Good luck ⌣̈

[5] 1. (a) Find the general solution of the PDE

$$y\partial_x u - x\partial_y u = 0. \tag{1}$$

[5] (b) Let R be the rectangle $\{(x, y) \mid x \in (1, 2)\,,\ y \in (1, 2)\}$. Suppose u solves the PDE (1) in R, and u is continuous up to the boundary of R. Must u attain it's maximum on the sides, bottom or top of the rectangle R? Explain.

[5] (c) Do the previous subpart for the rectangle rectangle $\{(x, y) \mid x \in (-1, 1)\,,\ y \in (-1, 1)\}$.

[10] 2. Solve the PDE $\partial_t u - \frac{1}{2}\partial_{xx} u = -u$ when $x \in \mathbb{R}$, $t > 0$ and $u(x, 0) = f(x)$. [Your answer should express $u(x, t)$ in terms of f, x, and t. It can involve nasty integrals. It might help to consider the function $v(x, t) = e^{\pm t}u(x, t)$.]

[10] 3. Let f be some given function. Suppose u solves the PDE $u_{tt} + u_{xt} - 2u_{xx} = f$ for $x \in \mathbb{R}$, $t > 0$ with initial conditions $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$. If $x_0 \in \mathbb{R}$, $t_0 > 0$ find a formula for $u(x_0, t_0)$ in terms of f, ϕ, ψ. [Hint: Any of the methods from class/the book to solve the inhomogeneous wave equation can be adapted to solve this equation. While you're free to use whatever method you like best, I only give you a hint about my 'favourite' one: Let Δ be the triangle with vertices (x_0, t_0), $(x_0 - 2t_0, 0)$ and $(x_0 + t_0, 0)$, and let D be the interior of this triangle. Compute $\int_\Delta 2u_x\, dt + (u_x + u_t)\, dx$ in two different ways: explicitly, and using Greens / Stokes theorems.]

[10] 4. (a) Suppose u solves the PDE $u_{tt} - u_{xx} = 0$ in the rectangle $x \in (0, \pi)$, $t \in (0, 1)$ with $u = 0$ on the sides, bottom and top of this rectangle. Show that u is the constant function 0. [Hint: Use separation of variables. You can assume that u is a continuous function.]

[5] (b) If we change $t \in (0, 1)$ to $t \in (0, \pi)$ above, then there are *infinitely many* non-zero solutions of the above PDE. Find one of them. [Namely, find a function u such that $u_{tt} - u_{xx} = 0$ in the rectangle $x \in (0, \pi)$, $t \in (0, \pi)$ with $u = 0$ on the sides, bottom and top of this rectangle, but u is not identically the 0 function.]

5. Let $f(x) = x^2$.

[5] (a) Using the convergence theorems from class/book what can you guarantee about pointwise, uniform or L^2 convergence of the Fourier cosine series of f on the interval $(0, \pi)$? Explain.

[5] (b) Compute the Fourier cosine series of f on the interval $[0, \pi]$.

[5] (c) Explicitly show that the Fourier cosine series of f converges uniformly on the interval $[0, \pi]$. (When I say 'explicitly', I mean do this directly from the answer in the previous part, *without* using the Fourier series convergence theorems from class.) [Hint: If A_n is the n^{th} Fourier coefficient of f, verify (using the previous part) that $\sum_1^\infty (nA_n)^2 < \infty$. Now do something similar to the proof I gave you of uniform convergence of Fourier series.]

[5] (d) Will the Fourier *sine* series of f converge uniformly to f on the interval $[0, \pi]$? [Hint: This one is short, easy, and does not rely on the evil previous part.]

[10] 6. Let D be some region in $\mathbb{R}^2$. Suppose u satisfies the PDE $\partial^2_{xx} u + 4\partial^2_{yy} u = 0$ in D, and is continuous up to the boundary of D. Show that u can attain a maximum on the interior of D only if u is identically a constant. [Hint: Don't reinvent the wheel. Do something 'clever' so that the strong maximum principle which I proved in class will apply.]

The extra credit is *harder* than the other questions. Please do not try it unless you are sure you have answered all other questions correctly and checked your work throughly. To dissuade you from prematurely trying the extra credit, I print it inconspicuously on the reverse. Further I will not offer any partial credit on it, and I do not tell you how much it is worth.

$\boxed{\star}$ 7. Let f be a function, and C be a constant such that for any θ_1, θ_2, $|f(\theta_1) - f(\theta_2)| \leqslant C|\theta_1 - \theta_2|$ (such a function is called a Lipshitz continuous function). Define u on a ball of radius 1 by the Poisson formula:

$$u(r,\theta) = \frac{1}{2\pi}\int P(r,\theta-\phi) f(\phi)\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} f(\phi)\,d\phi.$$

Show that

$$\lim_{r\to 1^-} \max_{\theta\in[0,2\pi)} |u(r,\theta) - f(\theta)| = 0$$

That is, show that u converges to f *uniformly* as $r \to 1^-$. [NOTE: All one needs to assume about f is that it is uniformly continuous. I write Lipshitz in the problem only because it is easier to explain than uniform continuity. If your proof requires some property about the Poisson kernels, other than what was done in class (e.g. something about the integrals etc.), then you should prove it.]