Math 131 Final

Friday December 14^{th}, 2007

Time: 3 hours
Total: 75 points

This is a closed book test. You may use without proof any thing done in class, or in sections of the book included for this exam. Calculators, cell phones, etc. are not allowed. Cell phones won't help you either, but do not feel free to use them. Good luck ◡̈

$\boxed{5}$ 1. (a) Let u satisfy the PDE $au_x + bu_y + cu_z = 0$. (Here u is a function of three variables, x, y and z). Consider the curve $(x(t), y(t), z(t))$ where x, y, z are functions of t such that

$$\frac{1}{a}\frac{dx}{dt} = \frac{1}{b}\frac{dy}{dt} = \frac{1}{c}\frac{dz}{dt} = 1.$$

Show that u is constant along the curve $(x(t), y(t), z(t))$.

$\boxed{5}$ (b) Find the solution of the PDE $u_x + u_y + (1+z)u_z = 0$ with initial data $u(x, y, 0) = f(x, y)$.

$\boxed{5}$ 2. (a) Suppose f and g are two odd functions of x, and u solves the PDE $u_{tt} - c^2 u_{xx} = 0$ with initial data $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. Show that $u(0, t) = 0$ for all $t > 0$.

$\boxed{10}$ (b) Suppose u solves the wave equation $u_{tt} - u_{xx} = 0$ on the region $x, t > 0$, with boundary condition $u(0, t) = 0$ and initial data $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$. What portion of the initial data influences the value of $u(1, 2)$. [If you don't understand what this means, I will instead accept an exact formula for $u(1, 2)$ involving only f and g. Also note that in class I have *not* derived a formula for solutions to the wave equation with these boundary conditions! Thus if you know it, you can only use it if you derive it. If you don't know it (which is as it should be), then to derive the formula remember D'Alembert (which you can blindly use without proof), what I did for the heat equation with similar boundary conditions, and look at the previous subpart.]

$\boxed{10}$ 3. Let c be a positive function (i.e. $c(x, t) > 0$ for all x, t). Let R be the rectangle $[0, L] \times [0, T]$. Suppose u satisfies $u_t - u_{xx} + cu = 0$, on the interior of R, and is continuous on the boundary of R. Show that u attains it's maximum on the sides or bottom of R, *provided the maximum is strictly positive.*

4. A negative maximum however need not be attained on the sides or bottom of R. Here's a counterexample.

$\boxed{5}$ (a) Find all *separated* solutions to the PDE $v_t - v_{xx} + v = 0$ with boundary conditions $v(0, t) = 0 = v(L, t)$. Namely if $v_n(x, t) = X_n(x)T_n(t)$ solves the given PDE with boundary conditions, then find the functions X_n, and T_n.

$\boxed{5}$ (b) Let $f(x) = \sum_1^\infty A_n X_n(x)$, where X_n is from the previous subpart. Find the solution of the PDE $v_t - v_{xx} + v = 0$ with boundary conditions $v(0, t) = 0 = v(L, t)$ and initial data $v(x, 0) = f(x)$. [Hint: You may assume for this question that that infinite sums can be differentiated as you please. Your answer should express $v(x, t)$ as an infinite sum involving A_n, x, t.]

$\boxed{5}$ (c) Suppose $L = \ln 2$, and w solves $-w_{xx} + w = 0$ with $w(0) = w(L) = -1$. Find w.

$\boxed{10}$ (d) Suppose now that u satisfies the same PDE $u_t - u_{xx} + u = 0$, but with boundary conditions $u(0, t) = -1 = u(L, t)$, and initial data $u(x, 0) = -1$. Show that

$$\int_0^L |u(x,t) - w(x)|^2 \, dx \leqslant e^{-2(1+\frac{\pi^2}{L^2})t} \int_0^L |1 + w(x)|^2 \, dx.$$

[Hint: Set $v = u - w$. What PDE, and boundary conditions does v satisfy? Part (b) helps.]

$\boxed{5}$ (e) Suppose now that

$$\lim_{t\to\infty} \max_{x\in[0,L]} |u(x,t) - w(x)| = 0$$

(extra 5 points if you can either prove this, or solve this subpart without this assumption). Show that for T large enough, u does not attain it's maximum on the sides or bottom of the rectangle $[0, L] \times [0, T]$. [Hint: Draw w.]

$\boxed{10}$ 5. Recall first some notation: If $\bar{x} = (x_1, x_2) \in \mathbb{R}^2$, we use $|\bar{x}| = \sqrt{x_1^2 + x_2^2}$ to denote the length of the vector $\bar{x}$. We use $\bar{0} = (0,0) \in \mathbb{R}^2$ to denote the origin.

Let $D = \{\bar{x} \in \mathbb{R}^2 \mid |\bar{x}| \leqslant a\}$ be a disk of radius a. Let u be a function such that $\triangle u = 0$ on the interior of D, and $u(\bar{x}) \geqslant 0$ for all $\bar{x} \in D$. If $|\bar{x}| \leqslant r < a$, then show that

$$u(\bar{0})\frac{a-r}{a+r} \leqslant u(\bar{x}) \leqslant u(\bar{0})\frac{a+r}{a-r}.$$

[This is called the *Harnack Inequality*, and is a rather striking result. In words it says that if u is a positive harmonic function on a disk of radius a, then the oscillation on any smaller disk can be uniformly controlled! The hint is to use the Poisson formula.]

You will not be awarded any partial credit for extra credit questions. It's all or nothing.

6. *(Extra credit)* Here's a more general version of the maximum principle. Let a, b, c be three functions. Assume a and c are non-negative (i.e. $a(x,t) \geqslant 0$ and $c(x,t) \geqslant 0$ for all x, t). Let R be the rectangle $[0, L] \times [0, T]$. Assume that u satisfies the PDE $u_t - au_{xx} + bu_x + cu = 0$. Show that u attains it's maximum on the sides or bottom of R, *provided the maximum is non-negative.* [NOTE: This question is a generalization of Question 3. Needless to say if you solution to this question is perfect, you automatically get a perfect score on both this question and Question 3. If you're feeling brave, only turn in a solution to this question. The extra bu_x term makes no difference, but the fact that none of the inequalities are strict makes things a little harder!]

7. *(Extra credit)* Let u satisfy $\triangle u = 0$ when $|\bar{x}| < a$ and $u(\bar{x}) = f(\bar{x})$ when $|x| = a$. If f is continuous at the point (a, θ) (in polar coordinates), show directly using the Poisson formula that $\lim_{r \to a} u(r, \theta) = f(a, \theta)$.