Assignment 14: Assigned Wed 04/25. Due Wed 05/02

- 1. (a) Let  $\Gamma$  be a (piecewise)  $C^1$  closed curve (oriented counter clockwise) in the plane, enclosing an open set U. Show that  $\frac{1}{2} \oint_{\Gamma} -y \, dx + x \, dy = \operatorname{Area}(U)$ . [Notation: Given a function  $F = \begin{pmatrix} P \\ Q \end{pmatrix}$ , we define  $\int_{\Gamma} P \, dx + Q \, dy$  to be the line integral  $\int_{\Gamma} F \cdot dl$ .]
  - (b) Let P be a polygon (not necessarily convex) in  $\mathbb{R}^2$ . Suppose the vertices of P (ordered counter clockwise) have coordinates  $(x_1, y_1), \ldots, (x_N, y_N)$ . Show that

Area(P) = 
$$\frac{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \dots + (x_Ny_1 - x_1y_N)}{2}$$
.

[This is a standard result from coordinate geometry. The 'pure geometry' proof, however is a little tricky. If you knew that your polygon was convex, you could divide it into triangles based at the  $(x_1, y_1)$  and prove this formula quickly. If your polygon is not convex, then doing this won't divide it up into disjoint triangles! So this trick (at first sight) won't work. But the Green's theorem line integral trick works just fine.]

2. (a) Let  $U, V \subseteq \mathbb{R}^2$  be open,  $\varphi : U \to V$  be  $C^2$ , and  $G : V \to \mathbb{R}^2$  be  $C^1$ . Define  $F : U \to \mathbb{R}^2$  by  $F(x) = (D\varphi_x)^* (G \circ \varphi(x))$ . Show that

$$\partial_2 F_1 - \partial_1 F_2 = [(\partial_2 G_1 - \partial_1 G_2) \circ \varphi] \det(D\varphi)$$

[This will conclude the proof from class for Greens theorem, when the domain is a  $C^2$  image of a rectangle.]

(b) Suppose instead  $V \subseteq \mathbb{R}^3$ , and  $G: V \to \mathbb{R}^3$ , then show

$$\partial_2 F_1 - \partial_1 F_2 = \left[ (\nabla \times G) \circ \varphi \right] \cdot (\partial_1 \varphi \times \partial_2 \varphi)$$

- (c) Prove Stokes theorem when the surface is the image of a square. Explicitly, suppose  $S \subseteq \mathbb{R}^3$  is a surface who's boundary is the closed curve  $\Gamma$ . Suppose further there exists an injective (piece-wise)  $C^2$  function  $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$  which parametrizes the surface S preserving orientation: Namely, this means that if  $R = (0, 1) \times (0, 1)$  is the unit square in  $\mathbb{R}^2$ , then we assume  $\varphi(R) = S$ , and  $\varphi(\partial R) = \Gamma$ . Further, if  $\gamma(t)$  traverses  $\partial R$  counter-clockwise, then  $\varphi \circ \gamma$  traverses  $\Gamma$  counter clockwise and  $(\hat{n} \circ \varphi) \cdot \partial_1 \varphi \times \partial_2 \varphi > 0$ . (Here  $\hat{n}$  is the (given) normal vector to the surface S). Show that  $\oint_{\Gamma} G \cdot dl = \int_{S} (\nabla \times G) \cdot \hat{n} \, dS$ .
- 3. The fundamental theorem for line integrals says that the line integral of a gradient only depends on the two end points of the curve. We address the converse here.
  - (a) Let  $F : \mathbb{R}^2 \to \mathbb{R}^2$  be a continuous function. Show that the line integral of F over any curve only depends on the two end points of the curve if and only if the line integral of F over any closed curve is 0.
  - (b) Let  $F : \mathbb{R}^2 \to \mathbb{R}^2$  be a continuous function so that the line integral of F over any curve *only* depends on the two end points of the curve. Show

that there exists a  $C^1$  function  $V : \mathbb{R}^2 \to \mathbb{R}$  such that  $F = \nabla V$ . [HINT: Let  $V(x,y) = \int_{(0,0)}^{(x,y)} F \cdot dl$ , where the integral is taken over *any* curve joining (0,0) and (x,y). Note that this gives a well defined function V by our assumption on F. Now choose the curve  $\Gamma$  to start from (0,0) go straight (vertically) to (0,y) and then straight (horizontally) to (x,y). Using this, show  $\partial_x V = F_1$ .]

- (c) Does the previous part work if  $F : \mathbb{R}^3 \to \mathbb{R}^3$  instead? Justify.
- 4. We know from an old homework that all gradients have 0 curl (i.e.  $\nabla \times \nabla \varphi = 0$ ). We claim that the converse is true, provided your domain is nice.
  - (a) Suppose  $F : \mathbb{R}^3 \to \mathbb{R}^3$  is a  $C^1$  function such that  $\nabla \times F = 0$ , show that there exists a  $C^1$  function V such that  $F = \nabla V$ . [HINT: Problem 3?]
  - (b) Compute  $\oint_{\Gamma} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ , where  $\Gamma$  is a circle with center 0.
  - (c) Let  $U = \mathbb{R}^3 \{(x, y, z) \mid x = y = 0\}$ . Find a  $C^{\infty}$  function  $F : U \to \mathbb{R}^3$  such that  $\nabla \times F = 0$ , but there does not exist  $V \in C^1(U)$  such that  $F = \nabla V$ . [HINT: Part (b)?]
- 5. (Winding number) This last question constructs the winding number of a curve. Let  $\Gamma$  be a (piecewise)  $C^1$  closed curve in  $\mathbb{R}^2 - \{0\}$  with parametrization  $\gamma$ . We define the winding number of  $\Gamma$  to be

$$w(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} F \cdot dl, \quad \text{where } F(x,y) = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \end{pmatrix}$$

We claim that this line integral exactly counts the number of times the curve  $\Gamma$  winds around the origin. One thing that might tip you off to this fact is that  $F = \nabla \tan^{-1}(y/x)$ , when  $x \neq 0$ , and so the line integral  $\int_{\Delta} F \cdot dl$  over a curve joining points P and Q should morally be  $\tan^{-1}(P) - \tan^{-1}(Q)$ , the 'angle' swept out by the curve.

This is correct, provided  $x \neq 0$ ! If the curve crosses the y axis, then our fundamental theorem doesn't apply, and we need to look elsewhere for a rigorous treatment.

(a) (Independent of the other subparts) Suppose the curve  $\Gamma$  is the boundary of an open set U that does not contain the origin. We should intuitively expect that  $w(\Gamma) = 0$ . Prove this. [HINT: Greens theorem...]

(b) Let 
$$\theta(t) = \int_0^t F(\gamma(t)) \cdot \gamma'(t) dt$$
. Show that  

$$\frac{d}{dt} \left[ |\gamma| \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \cdot F(\gamma) \right] = 0 = \frac{d}{dt} \left[ |\gamma| \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \cdot F(\gamma) \right]$$

(c) Show that  $w(\Gamma)$  is an integer for any closed curve  $\Gamma$ . [This follows quickly from the previous subpart. There are many deep (and sometimes surprising) applications of the "simple" fact that the winding number is an integer. One example is the Fundamental Theorem of Algebra, which I will do in class later. Another, is the Jordan Curve theorem which I won't have time to do.]