Assignment 13: Assigned Wed 04/18. Due Wed 04/25

1. **Sec. 4.5.** 3, 15
2. **Sec. 4.10.** 4, 8
3. Let $D \subseteq \mathbb{R}^2$ be the region bounded by the lines $y = x$, $y = x + 2\pi$ and $x = 0$. Let $f(x, y) = \sin(y - x)$.
   (a) Does Fubini's theorem apply when computing $\iint_D f(x, y)\,dx\,dy$?
   (b) Verify both the iterated integrals associated with $\iint_D f(x, y)\,dx\,dy$ are finite, but not equal.
4. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1, and suppose for simplicity $\varphi(0, 0) = (0, 0)$. For $\varepsilon > 0$, let P_ε be the parallelogram with vertices $(0, 0)$, $\varphi(\varepsilon, 0)$, $\varphi(0, \varepsilon)$ and $\varphi(\varepsilon, 0) + \varphi(0, \varepsilon)$. Compute $\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2} \operatorname{Area}(P_\varepsilon)$. [The real reason behind the change of variable formula is that $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \operatorname{Area}(\varphi(S_\varepsilon)) = |\det(D\varphi)|$, where S_ε is the square with diagonal points $(0, 0)$ and $(\varepsilon, \varepsilon)$. This, however, is a little harder to see, mainly because $\varphi(S_\varepsilon)$ need not have a "nice shape". If we approximate $\varphi(S_\varepsilon)$ by a parallelogram then computing the limit can be done directly, and is exactly the content of this question.]
5. (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be linear, and P be the parallelogram with sides $T(e_1)$ and $T(e_2)$. Find a formula for the area of P in terms of the matrix of T.
   (b) Let $U \subseteq \mathbb{R}^2$ be open, and $\varphi : U \to \mathbb{R}^3$ be C^1 and injective. Then $S \stackrel{\text{def}}{=} \varphi(U)$, the image of U under φ is a surface in $\mathbb{R}^3$. Guess a formula for the area of S in terms of φ. [Hint: Use the same intuition from change of variable: If P is a really small square in U, what is the area of $\varphi(P)$? The previous subpart helps.]
   (c) Let $f : U \to \mathbb{R}$ be C^1. Use your formula from the previous subpart to show that
   $$\operatorname{Area}\{(x, y, f(x, y)) \mid (x, y) \in U\} = \iint_U \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}\,dx\,dy.$$
   (d) Using your formula, to compute the surface area of a sphere of radius r.
6. Let $f : [a, b] \to (0, \infty)$ be differentiable, and $S \subseteq \mathbb{R}^3$ be the surface formed by rotating the graph of f about the x-axis. Explicitly,
   $$S = \{(x, y, z) \mid x \in [a, b] \text{ and } y^2 + z^2 = f(x)^2\}.$$
   Show that that the area of S is $\int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx$. [You might have already seen this formula from one variable calculus. You can derive it here using Fubini's theorem and the surface area formula from the previous question.]