Assignment 8: Assigned Wed 03/07. Due Wed 03/21

1. Prove the chain rule. [In class, my proof was very "imprecise". I often said "when h is small enough", etc. Replace all these with precise ε-δ statements, and write a complete rigorous proof.]
2. **Sec. 1.8.** 2, 10, 12.
3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable, $c \in \mathbb{R}$ be given andand define

$$u(x,t) = \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy \right) ds$$

Compute $\partial_t \partial_t u - c^2 \partial_x \partial_x u$.

Assignment 9: Assigned Wed 03/21. Due Wed 03/28

1. **Sec. 1.9.** 1.
2. **Sec. 3.3.** 9.
3. **Sec. 3.4.** 1.
4. In the last homework, we saw the gradient, divergence and curl operators. Notice we can combine these operators to form a few second order operators: e.g. $\nabla \cdot (\nabla f)$. There are of course 9 combinations you can formally write down, but not all of them make sense.
   (a) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function. Which of the 9 second order combinations of divergence, gradient and curl make sense?
   (b) Of the combinations that make sense, exactly one must *always* be 0. Which one?
   (c) Now let $u : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 function. Which of the 9 second order combinations of divergence, gradient and curl make sense?
   (d) Again, of the combinations that make sense, exactly one must *always* be 0. Which one?
   (e) Let $u : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^2 function. Show that $\nabla \times \nabla \times u = -\triangle u + \nabla \nabla \cdot u$. Here $\triangle u$ is called the Laplacian of u, and defined to be the column vector $(\nabla \cdot \nabla u_1, \nabla \cdot \nabla u_2, \nabla \cdot \nabla u_3)^*$. [In fact, for a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$, the Laplacian of f (denoted by the same symbol $\triangle f$) is defined to be $\nabla \cdot \nabla f$.]

Assignment 10: Assigned Wed 03/28. Due Wed 04/04

1. Fill in the following detail from the proof from Wed 03/28: How small should $\varepsilon > 0$ be so that the condition $|\partial_i \partial_j f(\xi) - \partial_i \partial_j f(a)| \leqslant \varepsilon$ for all $i, j \in \{1, \dots, n\}$ will guarantee

$$\Big| \sum_{i=1}^n \sum_{j=1}^n \partial_i \partial_j f(\xi) - \partial_i \partial_j f(a)(x_i - a_i)(x_j - a_j) \Big| \leqslant \frac{c}{2} |x-a|^2 ?$$

2. Suppose $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$.
   (a) Show that A is positive definite and non-degenerate if and only if $a > 0$ and $ac - b^2 > 0$.
   (b) What is the analogous condition for A to be negative definite and non-degenerate? [Warning: Don't blindly change signs!]
   (c) Using the previous two subparts, state sufficient conditions for a function $f : \mathbb{R}^2 \to \mathbb{R}$ to attain a (strict) local maximum at a.
   (d) Guess a condition (using determinants of minors) that will guarantee an $n \times n$ symmetric matrix is positive definite. What is the condition guarantying the matrix is negative definite? (No proofs are required for this part.)
3. Let $f : U \to \mathbb{R}$ be C^2, and $a \in U$. Suppose $Df_a = 0$, and the Hessian $H \stackrel{\text{def}}{=} (\partial_i \partial_j f(a))$ has at least one strictly positive and at least on strictly negative eigenvalue. Show that f has a saddle at a.
4. **Sec. 3.6.** 2, 7, 8.
5. Let A be an $n \times n$ matrix, not necessarily symmetric, such that for all $v \in \mathbb{R}^n$ with $v \neq 0$, we have $(Av) \cdot v < 0$. Show that there exists $c > 0$ such that for all $v \in \mathbb{R}^n$, we have $(Av) \cdot v \leqslant -c|v|^2$. [Hint: This doesn't involve any linear algebra! I claim it can be done using a 'clever' argument using the extreme value theorem.]
6. *(Optional, hard)* Find a twice differentiable function with a critical point at a who's Hessian at a is symmetric, negative definite and non-degenerate, however f does not attain a local maximum at a.