Math 269: Homework.

The problem numbers refer to problems from your text book. I will often assign problems which are not in the text book. *Keep in mind that there is a firm 'no late homework' policy.*

Assignment 1: Assigned Fri 01/20. Due Wed 01/25

1. **Sec. 1.4.** 5, 10.
2. If $x, y \in \mathbb{R}^n$, show that $||x| - |y|| \leqslant |x - y|$.
3. If $A = (a_{i,j})$ is an $m \times n$ matrix, define $|A| = \left(\sum_{i=1}^m \sum_{j=1}^n a_{i,j}^2\right)^{1/2}$.
   (a) If A is a $m \times n$ matrix, and $b \in \mathbb{R}^n$, then show that $|Ab| \leqslant |A||b|$.
   (b) More generally, if B is a $n \times p$ matrix, show that $|AB| \leqslant |A||B|$.
4. If $x, y, z \in \mathbb{R}^3$, show that $x \cdot (y \times z) = \det(x, y, z)$. Here $\det(x, y, z)$ denotes the determinant of the matrix with columns x, y, and z respectively. [In your recitation you'll probably see that $|x \cdot (y \times z)|$ is the volume of the parallelepiped with sides x, y, z. This is just geometric intuition I'd like you to know. Verify the above identity algebraically, without any geometry.]
5. **Sec. 1.5.** 5
6. Show that an infinite intersection of open sets is not necessarily open. [Taking complements, you'll see that an infinite union of closed sets need not be closed.]

⋆ 7. *(Hard, optional, challenge.)* If $S \subseteq \mathbb{R}$ is both open and closed, show that $S = \emptyset$ or $S = \mathbb{R}$. [You'll need the notion of *infimum* or *supremum*. So if you haven't heard those words, look them up before trying this problem.]

Assignment 2: Assigned Wed 01/25. Due Wed 02/01

1. Here are two possible definitions of the boundary of a set.
   (a) *(Class)* $\partial S = \overline{S} - \mathring{S}$.
   (b) *(Text book)* $\partial S = \{x \in \mathbb{R}^n \mid \forall r > 0, B(x,r) \cap S \neq \emptyset, \ \& \ B(x,r) \cap S^{c} \neq \emptyset\}$.

   Show that these are equivalent.
2. Show that ∂S is closed, directly using the text-book definition of boundary. [We did in class using the other definition. Since the two definitions are equivalent, you know that ∂S is certainly closed. Doing it directly using the other definition is mainly practice in using the definition.]
3. Let (a_n) be a sequence in $\mathbb{R}^d$, (c_n) a sequence in $\mathbb{R}$. Suppose $(a_n) \to \alpha$, and $(c_n) \to \gamma$ with $\gamma \neq 0$. Show that $(\frac{1}{c_n} a_n) \to \frac{1}{\gamma}\alpha$.
4. (a) If f is continuous at a, and $(a_n) \to a$, show that $(f(a_n)) \to f(a)$.
   (b) If for every sequence $(a_n) \to a$, we have $(f(a_n)) \to f(a)$ then show that f is continuous at a.
5. **Sec. 1.5.** 13, 14, 18, 23.

Assignment 3: Assigned Wed 02/01. Due Wed 02/08

1. Instead of considering iterated limits, we could consider limits of a function along arbitrary lines approaching the point in question. It has pitfalls similar to iterated limits.
   (a) Let $f : \mathbb{R}^d \to \mathbb{R}$, and suppose $\lim_{x \to a} f(x) = l$. Let $v \in \mathbb{R}^d$ be non-zero. Then show that $\lim_{t \to 0} f(a + tv)$ exists and equals l. [This is called a directional limit.]

   Conversely, suppose for all $v \in \mathbb{R}^2$, nonzero, the limits $\lim_{t \to 0} f(a + tv)$ all exist and are equal. It turns out that $\lim_{x \to a} f(x)$ need not exist. Here's an example.

   (b) Define the function $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = 1$ if $|y| \geqslant x^2$ or if $y = 0$, and $f(x, y) = 0$ otherwise. For every $v \in \mathbb{R}^2$ non-zero, show that the all directional limits of f at at the point $(0, 0)$ exist and are equal. Show however that the (full) limit $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist.

   [Iterated and directional limits give easy tests to check whether (full) limits exist. If the iterated limits are not equal, or if two directional limits are not equal, then the (full) limit can not exist. But be warned: if all directional limits are equal, and/or if the iterated limits are equal, it need not mean the full limit exists!]
2. **Sec. 1.5.** 21, 24, 10.
3. If the series $\sum |a_n|$ is convergent, we know that that the series $\sum a_n$ is convergent. Show that $|\sum_1^\infty a_n| \leqslant \sum_1^\infty |a_n|$.
4. Let A be an $n \times n$ matrix. We know if $|A| < 1$, then the series $\sum A^n$ is convergent. For 1×1 matrices (a.k.a real numbers), the converse is also true. However, for $n \times n$ matrices there are plenty of examples where $|A| > 1$ and the geometric series $\sum A^n$ is convergent. For any $R \in \mathbb{R}$, find a 2×2 matrix A such that $|A| > R$ *and* $\sum_1^\infty A^n$ is convergent.