Supplement: Continuity of composition.

Proposition 1. *Let $D_1 \subseteq \mathbb{R}^m$ be open, $D_2 \subseteq \mathbb{R}^{m'}$ be open, $f : D_1 \to \mathbb{R}^n$, and $g : D_2 \to \mathbb{R}^m$ be two functions. Suppose $a \in D_2$ and g is continuous at a. Suppose further $g(a) \in D_1$, and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

Let's do some scratch. Our task is to make $f(g(x))$ close to $f(g(a))$ by making x close enough to a. Let $y = g(x)$. If y is close enough to $g(a)$, then continuity of f guarantees that $f(y)$ will be close to $f(g(a))$, as desired. However, since g is continuous at a, making x close enough to a will ensure $g(x)$ is close to $g(a)$! Combining these two ideas should give the proof.

Proof. Let $\varepsilon > 0$ be arbitrary. First, since f is continuous at $g(a)$, we know

$$\exists \delta_1 > 0 \ni |y - g(a)| < \delta_1 \implies |f(y) - f(g(a))| < \varepsilon. \tag{1}$$

Now, since g is continuous at a, we know

$$\exists \delta_2 > 0 \ni |g(x) - g(a)| < \delta_2 \implies |f(g(x)) - f(g(a))| < \boldsymbol{\delta_1}. \tag{2}$$

Notice equation (2) has a big bold δ_1 on the right, and *not* some function of ε, as is usually the case. It's not a typo. It's the correct logical step, as explained in the paragraph preceding this proof.

Now, choose $\delta = \delta_2$. Then

$$\begin{aligned} |x - a| < \delta &\implies |g(x) - g(a)| < \delta_1 && \text{[by equation (2)]},\\ &\implies |f(g(x)) - f(g(a))| < \varepsilon && \text{[by equation (1)]}, \end{aligned}$$

finishing the proof! ☐

As mentioned in class, the point of this proposition is that continuous functions can be passed in and out of limits. Namely:

$$\left.\begin{aligned} f \text{ continuous at } l\\ \textit{and} \lim_{x\to a} g(x) = l \end{aligned}\right\} \implies \lim_{x\to a} f(g(x)) = f(l) = f\left(\lim_{x\to a} g(x)\right)$$