Homework Assignment 5

Assigned Wed 11/17. Due Fri 12/03.

1. *(Ornstein-Uhlenbeck process)* Find an explicit solution of the SDE

$$dX_t = \mu X_t \, dt + \sigma \, dW_t$$

where $\mu, \sigma \in \mathbb{R}$, and W is a 1D Wiener process. Also compute EX_t and $\operatorname{Var}(X_t)$.

2. (a) *(Brownian bridge)* Let $a, b \in \mathbb{R}$, W be a 1D Wiener process. Show that strong existence and uniqueness holds for the SDE

$$dX_t = \frac{b - X_t}{1-t} \, dt + dW_t; \quad t \in [0,1), X_0 = a.$$

Show that $\lim_{t \to 1^-} X_t = b$ almost surely. [This is called a Brownian bridge from a to b.]

(b) For any $T \in (0,1]$, are the laws of $\{X_t\}_{t \leqslant T}$ and $\{W_t\}_{t \leqslant T}$ absolutely continuous? Prove it. Also, in the case the laws are absolutely continuous, find the Radon Nikodym derivative.

3. Let $d \in \mathbb{N}$, $b : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^d$ be bounded, Borel measurable and $\sigma : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}^{d^2}$ be bounded and uniformly Lipschitz. Suppose further there exists $\lambda > 0$ such that for all $t \geqslant 0$ and $x, y \in \mathbb{R}^d$ we have

$$\sum_{i,j,k} \sigma_t^{(i,k)}(x) \sigma_t^{(j,k)}(x) y^{(i)} y^{(j)} = |\sigma_t(x)^* y|^2 \geqslant \lambda |y|^2.$$

Then prove weak existence and uniqueness for the SDE

$$dX_t = b_t(X_t) \, dt + \sigma_t(X_t) dW_t$$

for any given initial distribution μ.

4. Let b, σ be uniformly Lipschitz and bounded. Let ξ be a random variable with $E\xi^2 < \infty$. Define $X^{(0)} = \xi$, and

$$X_t^{(n+1)} = \xi + \int_0^t b_s(X_s^{(n)}) \, ds + \int_0^t \sigma_s(X^{(n)}) \, dW_s$$

(a) Show that $\lim_{M \to \infty} \sup_n P\left(|X_0^{(n)}| > |M|\right) = 0$ and for any $T > 0$,

$$\lim_{\delta \to 0^+} \sup_{n \in \mathbb{N}} P\left(\sup_{\substack{|t-s|<\delta, \\ s,t \in [0,T]}} \left| X_s^{(n)} - X_t^{(n)} \right| > \varepsilon \right) = 0$$

(b) Show that $(X^{(n)}) \xrightarrow{d} X$, where X is the unique, strong solution to $dX_t = b_t(X_t) \, dt + \sigma_t(X_t) \, dW_t$ with initial data ξ.

5. Let b, σ be uniformly Lipschitz functions on $\mathbb{R}^d$, and X be the (unique, strong) solution of the SDE $dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t$ with initial data $X_0 = x$.

(a) Show that $\lim_{t \to 0^+} \frac{1}{t} E(X_t - x) = b(x)$ and $\lim_{t \to 0^+} \frac{1}{t} E(X_t^{(i)} - x^{(i)})(X_t^{(j)} - x^{(j)}) = \sum_k \sigma_{ik}(x) \sigma_{jk}(x)$.

(b) Show that for all $\varepsilon > 0$, $\lim_{t \to 0^+} \frac{1}{t} P(|X_t - x| > \varepsilon) = 0$.