Homework Assignment 4

Assigned Wed 10/27. Due Wed 11/17.

1. (a) Let Ω be a complete metric space, and (μ_n) be a sequence of Borel probability measures on Ω. If $(\mu_n) \xrightarrow{\text{w}} \mu$, show that $\limsup \mu_n(K) \geqslant \mu(K)$ for all compact sets $K \subseteq \Omega$.

   (b) Let Ω be a Polish space, and $\mathcal{A}$ be a family of Borel probability measures on Ω. If every sequence in A has a weakly convergent subsequence, show that A is tight. [This is the converse to Prohorov's theorem. Usually the other direction is much more useful, and if you're unfamiliar with it look it up (see Billingsley for instance).]

2. Let $M \in \mathcal{M}_c^2$. Show that $\mathcal{L}_0$ is dense in $\mathcal{L}^*(M)$ (without assuming $\langle M \rangle$ is absolutely continuous).

3. *(Tanaka's formula and local time)* Let W be a standard 1D Brownian motion, and define

$$L_t^\varepsilon = \frac{1}{2\varepsilon} \lambda \left\{ s \in [0,t] \mid |W_s| \leqslant \varepsilon \right\}$$

   where λ denotes the Lebesgue measure. One would naturally expect that $\lim_{\varepsilon \to 0^+} L_t^\varepsilon$ measures the amount of time Brownian motion spends at 0. This problem proves the existence of limit.

   (a) Let f_ε be the (unique) function such that $f_\varepsilon(0) = f'_\varepsilon(0) = 0$, and $f''_\varepsilon = \frac{1}{\varepsilon} \chi_{[-\varepsilon,\varepsilon]}$. Show that

$$f_\varepsilon(W_t) - f_\varepsilon(0) = \int_0^t f'_\varepsilon(W_s)\, dW_s + L_t^\varepsilon$$

   (b) As $\varepsilon \to 0$, show that $E \int_0^t |f'_\varepsilon(W_s) - \operatorname{sign}(W_s)|^2 \, ds \to 0$.

   (c) As $\varepsilon \to 0$, show that $f_\varepsilon(0) \to 0$, and $E|f_\varepsilon(W_t) - |W_t||^2 \to 0$.

   (d) Conclude there exists an adapted, square integrable process L such that $E|L_t^\varepsilon - L_t|^2 \to 0$ as $\varepsilon \to 0$. Further show

$$|W_t| = \int_0^t \operatorname{sign}(W_s)\, dW_s + L_t$$

   This is called Tanaka's formula. [Remark: If we set $f(x) = |x|$, then $f'(x) = \operatorname{sign}(x)$, and $f''(x) = 2\delta_0$. Thus if we formally apply Itô's formula to $f(W)$, we exactly arrive at Tanaka's formula. Of course, since $f \notin C^2$, we may not apply Itô, and thus we have to resort to the approximations outlined above.]

4. *(Bessel processes)* Let $d > 1$, and W be a standard d-dimensional Brownian motion. Let $R = |W|$.

   (a) Let $B = \sum_1^d \int_0^t \frac{W_s^{(i)}}{R_s} \, dW_s^{(i)}$. Show that B is a standard 1D Brownian motion.

   (b) Show that $dR_t = \dfrac{d-1}{2R_t} \, dt + dB_t$. [Remark: Tanaka's formula shows that for $d = 1$, this equation does not hold. The Bessel process can be used to study questions about the return of Brownian motion to the origin. We know for $d = 1$, $P(|W_t| > 0 \; \forall t > 0\} = 0$. However, for $d > 1$, $P(|W_t| > 0 \; \forall t > 0) = 1$.

   It turns out that for $d = 2$, while Brownian motion does not return to the origin, it does come arbitrarily close. However, for dimensions 3 and higher, once Brownian motion leaves the origin, it remains bounded away from the origin almost surely. Details of the proof of this can be found in Karatzas and Shreve §3.3C]