Homework Assignment 3

Assigned Wed 10/06. Due Wed 10/27.

Questions 2(a) and 3(c) are a little harder, and you can probably find a hint an almost any book on Stochastic Calculus, as they're standard results. However I recommend you try them on your own first.

1. Let $\mathcal{F}$ denote the Borel σ-algebra on the Wiener space $C[0,\infty)^d$, and let P be the Wiener measure. As we did in class, for any $x \in \mathbb{R}$, $F \in \mathcal{F}$, define $P^x(F) = P(F - x)$. Show that the function $x \mapsto P^x(F)$ is Borel measurable.

2. For any $\lambda > 0$, we define the *Poisson process with intensity* λ as follows. Let τ_i be a sequence of i.i.d exponential random variables with parameter λ (i.e. $P(\tau_i \in dt) = \frac{1}{\lambda} e^{-\lambda t}$). Let $\sigma_n = \sum_1^n \tau_i$. The intuition is that τ_i is the time at which the i^{th} customer arrives, and σ_n is the time it takes for the first n customers to arrive.

   Now we define $N_t = \max\{n \in \mathbb{N} \mid \sigma_n \leqslant t\}$ to be the number of customers who've arrived up to time t. This is called the Poisson process with intensity λ.

   (a) Show that $N_t - N_s$ is an integer valued Poisson random variable with parameter $\lambda(t-s)$. Further, show $N_t - N_s$ is independent of $\mathcal{F}_s^N$.

   (b) Show that $\langle N \rangle_t = \lambda t$ almost surely.

   (c) Show that N is a Markov process.

3. (a) Let B be a Brownian motion. Find a Borel function f such that the process $\{f(B_t)\}_{t \geqslant 0}$ is not a Markov process.

   (b) Show that the process $X_t = |B_t|$ is a Markov process. Further show that the transition density $p_+(h,x,y) \stackrel{\text{def}}{=} P^0(X_{t+h} \in dy \mid X_t = x) = p(h,x,y) + p(h,x,-y)$, where p is the transition density of Brownian motion, and $x, y \geqslant 0$.

   (c) Let $Y_t = M_t - B_t$, where $M_t = \sup_{s \leqslant t} B_s$. Show that Y is a Markov process, with the same transition density as X. Conclude that X and Y have the same finite dimensional distributions.