Homework Assignment 2

Assigned Thu 09/16. Due Wed 10/06.

1. Say X_t is a process with independent increments. Show that if $s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s^X$. [Recall $\mathcal{F}_t^X = \sigma(\cup_{s \leqslant t} \sigma(X_s))$. Also, X has independent increments means that for any finite sequence $0 \leqslant t_0 < t_1 \cdots < t_n$, the family of random variables $\{X_{t_j} - X_{t_{j-1}} \mid 1 \leqslant j \leqslant n\}$ is independent.]

2. Let $B = \{(B^{(1)}, \dots, B^{(d)}), \mathcal{F}_t\}$ be a d dimensional Brownian motion starting at 0. Show that each coordinate $B^{(i)} \in \mathcal{M}_c^2$, and that $\langle B^{(i)}, B^{(j)} \rangle_t = \delta_{ij} t$.

3. Let B be a standard 1D Brownian motion. Show that for $\alpha > \frac{1}{2}$, $\lim_{t \to \infty} \frac{|B_t|}{t^\alpha} = 0$ almost surely. [Don't use the law of iterated logarithm, as the proof relies on this problem.]

4. Let B be a standard 1D Brownian motion. Show that each of the following are also standard 1D Brownian motions:
   (a) $\{-B_t\}_{t \geqslant 0}$
   (b) $\{\frac{1}{\sqrt{\lambda}} B_{\lambda t}\}_{t \geqslant 0}$, for any $\lambda > 0$.
   (c) $W_t = t B_{\frac{1}{t}}$ for $t > 0$, and $W_0 = 0$.
   (d) $\{B_{s+t} - B_s\}_{t \geqslant 0}$ for any fixed $s \geqslant 0$.

5. Let B be a standard 1D Brownian motion. Show that for any $\alpha > \frac{1}{2}$, $t \geqslant 0$,

$$\limsup_{h \to 0^+} \frac{|B_{t+h} - B_t|}{h^\alpha} = \infty,$$

   almost surely. [Note that from the previous two problems, you immediately have for $\alpha < \frac{1}{2}$, $\lim_{h \to 0+} \frac{|B_{t+h} - B_t|}{h^\alpha} = 0$ almost surely.]