Homework Assignment 1

Assigned Mon 08/30. Due Mon 09/13.

1. Let σ, τ be two stopping times.
   (a) Show that $\sigma \wedge \tau$, $\sigma \vee \tau$, $\sigma + \tau$ are also stopping times.
   (b) If $\sigma \leqslant \tau$ almost surely, then show $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$.
2. (a) Let $\{\mathcal{F}_n \mid n \in \mathbb{N}\}$ be a *decreasing* sequence of σ-algebras (i.e. $\mathcal{F}_n \supseteq \mathcal{F}_{n+1}$), and $\{X_n, \mathcal{F}_n \mid n \in \mathbb{N}\}$ be a backward submartingale (i.e. $E(X_n|\mathcal{F}_{n+1}) \geqslant X_{n+1}$). If $\inf_{n\in\mathbb{N}} EX_n > -\infty$, then show that $\{X_n \mid n \in \mathbb{N}\}$ is uniformly integrable.
   (b) Let $\{X_t, \mathcal{F}_t\}$ be a right continuous submartingale. Show that the function $t \mapsto EX_t$ is right continuous.
3. Let $\{X_t, \mathcal{F}_t \mid 0 \leqslant t < \infty\}$ be a right continuous martingale. Show that the following are equivalent.
   (a) $\{X_t \mid 0 \leqslant t \leqslant \infty\}$ is a uniformly integrable family of random variables.
   (b) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $X_t \to X_\infty$ in L^1 as $t \to \infty$. (Recall $\mathcal{F}_\infty = \sigma(\cup_t \mathcal{F}_t)$.)
   (c) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $X_t \to X_\infty$ almost surely and $\{X_t, \mathcal{F}_t \mid 0 \leqslant t \leqslant \infty\}$ is a martingale.
   (d) There exists $X_\infty \in L^1(\Omega, \mathcal{F}_\infty)$ such that $\{X_t, \mathcal{F}_t \mid 0 \leqslant t \leqslant \infty\}$ is a martingale.
4. Let X be an integrable, progressively measurable process such that $EX_\tau = 0$ for all stopping times τ such that $P(\tau < \infty) = 1$. Show that X is a martingale.
5. Let X be a continuous local martingale. Show that there exists a localizing sequence (τ_n) such that for all n, the stopped process $X^{\tau_n} = \{X_{\tau_n \wedge t}, \mathcal{F}_t\}$ is bounded almost surely. (Note, we say (τ_n) is a localising sequence if it is an increasing sequence of stopping times which converges to ∞ almost surely, and for all n, the stopped process X^{τ_n} is a martingale).