Math 341 Syllabus and Lecture schedule.

Gautam Iyer, Fall 2010

L1, Mon 01/10.
- Introduction & motivation
- Fields
  - Definition.
  - Examples. $\mathbb{R}, \mathbb{Q}, \{0,1\}$, etc. (Also non-examples like $\mathbb{Z}$).
  - Multiplication by 0.

L2, Wed 01/12.
  - Uniqueness of inverses. Inverse of inverses.
  - $(-a)(-b) = ab$, etc.
  - Complex Numbers

L3, Fri 01/14.
- Vector spaces
  - *'Head-to-toe'* addition in $\mathbb{R}^2$.
  - Axiomatic definition.
  - Examples: $0, \mathbb{R}^n, F^n$, function spaces.

L4, Wed 01/19.
  - Subspaces, examples.
  - $U \subseteq V$ a subspace iff $U \neq \emptyset$ and $\forall u, v \in U$, $\alpha \in F$, $u + \alpha v \in U$.

L5, Fri 01/21.
  - Span, Linear Independence
    * Define $\operatorname{span}\{v_1, \dots, v_n\}$, and show it is a subspace.
    * Define linear independence, linear dependence, and Basis

L6, Mon 01/24.
    * Example of L.I.
    * Let S be L.I. Then $S \cup \{u\}$ is L.I. iff $u \notin \operatorname{span}(S)$.

L7, Wed 01/26.
  - Basis and Dimension
    * If V is spanned by n vectors, then any $n+1$ vectors in V are linearly dependent.
    * Linearly independent lists can't be longer than spanning lists.

L8, Fri 01/28.
    * Any two (finite) basis have the same cardinality.
    * Any L.I. subset of a finitely generated vector space can be extended to a basis.
    * Any finitely generated vector space has a basis.
    * Define dimension of a vector space.

L9, Mon 01/31.
    * Any subspace of a finitely generated vector space is also finitely generated; and hence has dimension not larger than that of the whole space.
    * Any n L.I. vectors in a space of dimension n are also spanning.
    * Any n spanning vectors in a space of dimension n are also L.I.

L10, Wed 02/02.
    * Any vector can be *uniquely* expressed as a linear combination of basis vectors.
  - Error correcting codes.
    * $F = \{0,1\}$, choose $a, b, c \in F$.
    * Pick $\{v_1, v_2, v_3\} \in F^n$, and transmit the message $u = av_1 + bv_2 + cv_3$.
    * Let $C = \operatorname{span}\{v_1, v_2, v_3\}$. If $e_i \notin C$ for all i, then *one* error can be detected.
    * If $e_i + e_j \notin C$ for $i \neq j$, then *one* error can be corrected.
    * E.g. $n = 6$, $v_1 = e_1 + e_2 + e_3$, $v_2 = e_1 + e_4 + e_5$, $v_3 = e_1 + e_3 + e_4 + e_6$. Allows you to transmit a 3 bit message using 6 bits, such that an error of at most 1 bit can be corrected!

L11, Fri 02/04.
  - $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$.

L12, Mon 02/07.
- Linear transformations
  - Definition, and some examples.
  - Kernel, image. Proof that $\ker(T)$ and $\operatorname{im}(T)$ are subspaces.
  - $\mathcal{L}(U, V)$, and proof that it is a vector space.

L13, Fri 02/11.
    * Closure of $\mathcal{L}(U, U)$ under composition.
    * Associativity, non-commutativity, Identity.

L14, Mon 02/14.
    * $T \in \mathcal{L}(U, V)$ bijective $\implies T^{-1} \in \mathcal{L}(V, U)$.
    * $T \in \mathcal{L}(U, V)$ injective $\iff \ker(T) = \{0\}$.

L15, Wed 02/16.
  - Rank Nullity: $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim(\operatorname{domain}(T))$.
    * If $T \in \mathcal{L}(U, V)$, and $\dim(U) < \dim(V) < \infty$ then T is not surjective.
    * If $T \in \mathcal{L}(U, V)$, and $\infty > \dim(U) > \dim(V)$ then T is not injective.
    * Any system of linear homogeneous equations with more variables than equations has a non-zero solution.

L16, Fri 02/18.
    * Isomorphisms.
  - Matrices
    * Linear transformations can be uniquely determined from values on basis vectors.
    * Define matrix representations of vectors and linear transformations.
    * Compute $\mathcal{M}_C(Tu)$ in terms of $\mathcal{M}_{B,C}(u)$ and $\mathcal{M}_B(u)$.

L17, Mon 02/21.
    * Define $\operatorname{Mat}(n, m, F)$
    * Motivate the definition of matrix multiplication by composition of linear transformations.

L18, Wed 02/23.
- Linear equations
  - Elementary row operations don't change solutions of $Ax = b$.

- Row reduced echelon matrices.
  - Pivots, free variables, and 'reading off' solutions to $Ax = b$.
  - *L19, Fri 02/25.* Row reduction to row reduced echelon form.
- Matrix inverses: If $\operatorname{rref}(A|I) = (I|B)$ then $B = A^{-1}$.

L20, Mon 02/28.
- Polynomials
  - Define $P_F(x)$ to be the ring of formal polynomials.
  - Degree, addition, multiplication.
  - For $p, q \in P_F(x)$, with $q \neq 0$, $\exists s, r \in P_F(x)$ such that $p = qs+r$ and $\deg(r) < \deg(q)$.
  - *L21, Wed 03/02.* The evaluation map E_α, roots, divisibility.
  - If α is a root of f then $(x - \alpha)|f$
  - Fundamental theorem of Algebra
  - If $f \in P_{\mathbb{C}}(x)$ is non-constant, then f factors completely as a product of linear polynomials.

L22, Mon 03/14.
- Eigenvalues
  - Definition
  - Eigenvalues of diagonal matrices.
  - Eigenvectors corresponding to distinct eigenvalues are linearly independent.
  - If $\dim(V) < \infty$, then λ an eigenvalue of $T \iff T - \lambda I$ is not invertible.
  - *L23, Wed 03/16.* If $F = \mathbb{C}$, $\dim(V) < \infty$, then every linear transformation from V to V has an eigenvalue.
  - Explicitly computing eigenvalues of 2×2 matrices.
  - *L24, Fri 03/18.* Tournament ranking / page rank.
  - *L25, Mon 03/21.* Diagonalizable transformations, matrices.
    - Computing powers of diagonalizable matrices.
    - Computing Fibonacci numbers as an application.
  - *L26, Fri 03/25.* Basis change: If B, C are two basis of V then
    - For $x \in V$, $\mathcal{M}_C(x) = \mathcal{M}_C(B)\mathcal{M}_B(x)$.
    - $\mathcal{M}_C(T) = P^{-1}\mathcal{M}_B(T)P$, where $P = \mathcal{M}_B(C)$.
  - *L27, Mon 03/28.* Upper triangular matrices.
    - If $F = C$, and $\dim(V) < \infty$, then any linear transformation has a basis under which it is upper triangular.
    - *L28, Wed 03/30.* Invertibility, and eigenvalues of upper triangular matrices.

L29, Fri 04/01.
- Inner product spaces
  - $\langle x, y\rangle = \sum_i x_i y_i = y^{\mathrm{t}} x$ as a generalization of $\|x\|\|y\| \cos\alpha$.
  - Using the inner-product to define length: $\|x\| = \sqrt{\langle x, x\rangle}$.
  - Complex inner products.
  - *L30, Mon 04/04.* Orthogonality, Pythagoras theorem.
  - Cauchy-Schwartz inequality
  - *L31, Wed 04/06.* Triangle inequality.
  - Orthonormal sets.
    - Linear independence.
    - Coordinates with respect to an orthonormal basis.
    - *L32, Fri 04/08.* Gram-Schmidt orthogonalization.
  - *L33, Mon 04/11.* Orthogonal projections.
    - Existence, and uniqueness.
    - Length minimizing property: If P is the orthogonal projection onto U, then $\|v - Pv\| \leqslant \|v - u\|$ for all $v \in V$, $u \in U$.
  - *L34, Wed 04/13.* Introduction to Least squares.
    - Minimize error when solving $Ax = b$
    - The *least squares* solution x_* is defined to be the solution of $Ax_* = Pb$, where P is the orthogonal projection onto $\operatorname{im}(A)$.
    - The least squares solution can be found by solving the normal equation $A^*Ax_* = A^*b$ (proof next week).
  - Orthogonal compliments.
    - If P is the orthogonal projection onto U, then $U = \operatorname{im}(P)$, and $U^\perp = \ker(P)$.
    - $\dim(U) + \dim(U^\perp) = \dim(V)$.
    - *L35, Mon 04/18.* For all $v \in V$, there exist unique $u_1 \in U$ and $u_2 \in U^\perp$ such that $v = u_1 + u_2$.
  - Duals
  - If $\dim(V) < \infty$, and $\varphi \in V^*$ then there exists a unique $v \in V$ such that $\varphi(u) = \langle u, v\rangle$ for all $u \in V$.
    - Proof 1: $T : V \to V^*$ by $T(v)(u) = \langle u, v\rangle$, and check that T is (conjugate) linear and injective. This would imply T is surjective, finishing the proof.
    - *L36, Wed 04/20.* Proof 2: Let $U = \ker(\varphi)$. If $U = V$, then choose $v = 0$. If not, pick any $v_1 \in V$ non-zero, and choose $v = \frac{\overline{\varphi(v_1)}}{\|v_1\|^2} v_1$.
  - Adjoints
    - If $V = \mathbb{R}^n$, with the standard inner product, $A \in \operatorname{Mat}(n, n, \mathbb{R})$, and $Tx = Ax$, then $T^*x = A^{\mathrm{t}}x$.
    - If $V = \mathbb{C}^n$, with the standard inner product, $A \in \operatorname{Mat}(n, n, \mathbb{C})$, and $Tx = Ax$, then $T^*x = \overline{A^{\mathrm{t}}}x$.
    - If $\dim(V) < \infty$, then T^* exists.
    - *L37, Fri 04/22.* Uniqueness of the adjoint.
  - Self-adjoint operators
    - $T = T^* \implies$ all eigenvalues are real.

    - $T = T^* \implies$ eigenvectors corresponding to distinct eigenvalues are orthogonal.
- Orthogonal transformations

L38, Mon 04/25.

- Normal operators
    - T normal $\implies T - \lambda I$ is also normal.
    - T normal $\implies \|Tv\| = \|T^*v\|$
    - T normal and $Tv = \lambda v \implies T^*v = \bar{\lambda}v$.
    - Spectral theorem: T normal, $\dim(V) < \infty$, $F = \mathbb{C}$, then there exists an orthonormal basis of V consisting of eigenvectors of T. (Proof: Choose an eigenvector v_1, put $U = \operatorname{span}\{v_1\}^\perp$, show $T, T^* \in \mathcal{L}(U, U)$, and use induction.)