Homework Assignment 9

Assigned Fri 04/01. Due Fri 04/08.

1. *(Optional)* Let M be an upper triangular matrix. Show that the eigenvalues of M are exactly the diagonal entries of M. [I stated, but did not finish proving this in class.]

2. (a) Let $M = \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix}$, and $N = \begin{pmatrix} \mu_1 & * & * \\ & \ddots & * \\ & & \mu_n \end{pmatrix}$. Show that MN is upper triangular. What are the diagonal entries of MN?

   (b) Let B be a basis of V and $T \in \mathcal{L}(V, V)$ be such that $\mathcal{M}_B(T)$ is upper triangular. Show that $\mathcal{M}_B(T^{-1})$ is also upper triangular.

Let $T \in \mathcal{L}(V, V)$. We say $f \in P_F(x)$ is the *minimal polynomial* of T if f has leading coefficient 1, and f is the polynomial of *smallest* degree such that $f(T) = 0$. That is, f is of the form $1 \cdot x^n + a_{n-1}x^{n-1} \cdots + a_0$, $f(T) = 0$, and whenever $g(T) = 0$, $\deg(g) \geqslant \deg(f)$.

3. (a) Suppose $T \in \mathcal{L}(V, V)$ is diagonalizable, and $\lambda_1, \ldots, \lambda_k$ are the k *distinct* eigenvalues of T. Show that $f = (x - \lambda_1) \cdots (x - \lambda_k)$ is the minimal polynomial of T. [Note: We know $k \leqslant \dim(V)$ from class, however we have not assumed $k = \dim(V)$.]

   (b) Compute the minimal polynomial of $\left(\begin{smallmatrix} -1 & 2 \\ -2 & 3 \end{smallmatrix}\right)$.

4. Here are two important properties about minimal polynomials.

   (a) Let f be the minimal polynomial of T. If $g \in P_F(x)$ is such that $g(T) = 0$, show that $f | g$.

   (b) If $P \in \mathcal{L}(V, V)$ is invertible, show that T and $P^{-1}TP$ have the same minimal polynomial.

5. You might wonder if any linear transformation has a minimal polynomial. This is guaranteed on *finite dimensional* vector spaces.

   (a) Suppose $\dim(V) = n < \infty$, and $T \in \mathcal{L}(V, V)$. Show that there exists a non-zero polynomial $g \in P_F(x)$ with $\deg(g) \leqslant n^2$ such that $g(T) = 0$. Conclude that T has a (unique) minimal polynomial of degree at most n^2. [On a later homework, we'll show that in fact the minimal polynomial can have degree at most n. This is the Cayley-Hamilton theorem.]

   (b) Let $V = P_F(x)$, and define $T \in \mathcal{L}(V, V)$ by $Tf = f'$. Show that T has no minimal polynomial.

6. We've seen previously that that the minimal polynomial of a diagonalizable linear transformation factors into a product of distinct linear factors. The converse is also true, and this question is devoted to proving it. [Note that any polynomial over $\mathbb{C}$ factors into linear factors, but these factors are not necessarily distinct! The key point in this statement is that a transformation is diagonalizable iff the minimal polynomial factors into *distinct* linear factors.]

   (a) Let $\lambda_1, \ldots, \lambda_k \in F$ be distinct. Let $p_i \in P_F(x)$ be defined by $p_i = \prod_{j \neq i} \frac{x - \lambda_j}{\lambda_i - \lambda_j}$. If $g \in P_F(x)$ with $\deg(g) < k$, then show that $g = \sum_1^k g(\lambda_i) p_i$. In particular, show $\sum p_i = 1$. [Hint: Show first $p_i(\lambda_j) = 1$ if $i = j$ and 0 otherwise. Now how many roots does $g - \sum g(\lambda_i) p_i$ have?]

   For the next few parts, let λ_i, p_i be as in the previous part. Let $T \in \mathcal{L}(V, V)$ be such that the minimal polynomial of T is $\prod_i (x - \lambda_i)$. Let $P_i = p_i(T)$, and $E_{\lambda_i} = \ker(T - \lambda_i I)$.

   (b) Show that for all i, $P_i \in L(V, E_i)$ and $P_i^2 = P_i$. Further, for all $i \neq j$ show that $P_i P_j = 0$.

   (c) Show that $\sum_{i=1}^k P_i = I$. Conclude that T is diagonalizable. [Hint: If B_i is a basis of E_{λ_i}, show that $B = \cup_i B_i$ is a basis of V.]