Homework Assignment 8

Assigned Fri 03/25. Due Fri 04/01.

1. Say $f = \sum_0^n a_i x^i \in P_F(x)$. Recall that for $T \in \mathcal{L}(V, V)$ we define $f(T) = \sum_0^n a_i T^i$, with the convention that $T^0 = I$.
   (a) In general if $S, T \in \mathcal{L}(V, V)$, show that you need not have $f(S)g(T) = g(T)f(S)$.
   (b) If however $ST = TS$, then show that $f(S)g(T) = g(T)f(S)$.
2. Let $T \in \mathcal{L}(V, V)$ and $f \in P_F(x)$. If λ is an eigenvalue of T, show that $f(\lambda)$ is an eigenvalue of $f(T)$. Hence, or otherwise, show that if $f(T) = 0$, then λ is a root of f.
3. Let $\lambda_1, \dots, \lambda_n \in F$, and M be the $n \times n$ diagonal matrix defined by

$$M = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

   Recall our convention that blank entries in a matrix are assumed to be 0. We've seen in class that $\lambda_1, \dots, \lambda_n$ are all eigenvalues of M. Show that M has no other eigenvalues.
4. In class, we saw that if a matrix M can be diagonalized, the powers M^n can be easily computed explicitly. When M can't be diagonalized, the story is a little more complicated. Here's an example.
   (a) Let $M = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$, with $1 \neq 0$. Find an explicit formula for M^n.
   (b) Let $F = \mathbb{R}$, and $A = \begin{pmatrix} 4 & 4 \\ -1 & 8 \end{pmatrix}$. Show that A is not diagonalizable. (I.e. show that there does not exist a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.)
   (c) Find $v_1, v_2 \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $Av_1 = \lambda v_1$, and $Av_2 = \lambda v_2 + v_1$.
   (d) Compute an explicit formula for A^n.
5. (a) *(Hard)* If $S, T \in \mathcal{L}(V, V)$ are such that $I - ST$ is invertible, then show that $I - TS$ is invertible.
   (b) If $\dim(V) < \infty$, and λ is an eigenvalue of ST, show that λ is an eigenvalue of TS.