Homework Assignment 4

Assigned Fri 01/28. Due Fri 02/04.

1. (a) Let $S = \{v_1, \dots, v_n\}$, and assume $v_1 \neq 0$. Show that S is linearly dependent, if and only if there exists $j \in \{2, \dots, n\}$ such that $v_j \in \operatorname{span}\{v_1, v_2, \dots, v_{j-1}\}$.

   (b) Suppose S is linearly dependent, and j is as in the previous subpart, show that $\operatorname{span}(S) = \operatorname{span}(S - \{v_j\})$ (i.e. show $\operatorname{span}\{v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_n\} = \operatorname{span}\{v_1, \dots, v_n\}$).

2. Suppose $V = \operatorname{span}\{v_1, \dots, v_n\}$, and $V \neq \{0\}$. Show that any (finite) spanning set contains a basis. Namely, if $\operatorname{span}\{u_1, \dots, u_m\} = V$, then show that there exists $\{w_1, \dots, w_k\} \subseteq \{u_1, \dots, u_m\}$ such that $\{w_1, \dots, w_k\}$ is a basis of V.

3. (a) Suppose $S = \{v_1, \dots, v_n\}$ is linearly independent. Show that any non-empty subset of S is also linearly independent.

   (b) Suppose $S = \{v_1, \dots, v_n\}$ is linearly dependent. Show that any finite superset of S is also linearly dependent.

4. Let V be a vector space over some field F, and $B \subseteq V$ a finite and non-empty set. Prove that the following statements are equivalent:

   (a) B is a basis of V.

   (b) B is a maximal linearly independent subset of V. Namely B is linearly independent, and if $C \supsetneq B$, then C is linearly dependent.

   (c) B is a minimal spanning set. Namely $\operatorname{span}(B) = V$, and if $C \subsetneq B$, then $\operatorname{span}(C) \subsetneq V$.

   [The usual way to prove that a list of statements are equivalent is to prove (a) implies (b), (b) implies (c) and finally (c) implies (a). While you're free to use your own favourite set of implications, I recommend following the above structure in this case.]

5. So far we've dealt with the spans of finite non-empty sets. Here's how we do it for infinite sets.

   (a) Let $S = \{v_1, \dots, v_n\} \subseteq V$, and $U = \operatorname{span}(S)$. Show that U is the *minimal* subspace of V that contains S. That is, show that $U \supseteq S$, and further, if U' is *any* subspace of V such that $U' \supseteq S$, then $U' \supseteq U$.

   Note that I used the word 'minimal' and not 'smallest' above. The reason for this is that subspaces are usually infinite sets, and comparing cardinalities won't give any useful information (e.g. $\mathbb{R}$ and $\mathbb{R}^2$ have the same cardinality).

   Now, let $S \subseteq V$ be any subset (not necessarily finite). We *define* $\operatorname{span}(S)$ to be the minimal subspace of V that contains S! Of course, before we can make such a definition, we must show that the "minimal subspace of V containing S" exists, and is unique!

   (b) Let U the intersection of all subspaces of V that contain S. Show that U is a *minimal* subspace of V that contains S. [This shows existence.]

   (c) If U and U' are two minimal subspaces of V that contain S, then show $U = U'$. [This shows uniqueness. Now, it is legitimate for us to define $\operatorname{span}(S)$ to be *the* minimal subspace of V containing S.]

   (d) Using the definition of span above, what is $\operatorname{span}(\emptyset)$?

   (e) If $S \neq \emptyset$, show that $\operatorname{span}(S) = \{\sum_{i=1}^{n} \alpha_i v_i \mid n \in \mathbb{N}, \alpha_i \in F, v_i \in S\}$. (That is, $\operatorname{span}(S)$ is the set of all *finite* linear combinations of elements in S.)

   Your next homework will contain a question describing linear independence of infinite sets.