Homework Assignment 11

Assigned Fri 04/22. Due Fri 04/29.

1. Let V be an inner product space.
   (a) If $U \subseteq V$ is a subspace, show that $U \subseteq (U^\perp)^\perp$.
   (b) If $\dim(V) < \infty$, show that $U = (U^\perp)^\perp$.

2. Suppose we want to find a line $y = c_0 + c_1 x$ that 'best fits' the points $(0,0)$, $(1,1)$, $(1,2)$.
   (a) If indeed there was a line that passed through the three points above, show that the coefficients c_0 and c_1 satisfy $A\left(\begin{smallmatrix} c_0 \\ c_1 \end{smallmatrix}\right) = \vec{b}$, where $\vec{b} = \left(\begin{smallmatrix} 0 \\ 1 \\ 2 \end{smallmatrix}\right)$, and $A = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.
   (b) With A and $\vec{b}$ as above, show that the system $A\vec{x} = \vec{b}$ has no solutions.
   (c) Let $\left(\begin{smallmatrix} c_0 \\ c_1 \end{smallmatrix}\right)$ be the least square solution to $A\vec{x} = \vec{b}$. Sketch the line $y = c_0 + c_1 x$ and the points $(0,0)$, $(1,1)$, $(1,2)$ on the same graph.

3. Let U, V be two finite dimensional inner product spaces, with inner products $\langle\cdot,\cdot\rangle_U$ and $\langle\cdot,\cdot\rangle_V$ respectively.
   (a) If $T \in \mathcal{L}(U,V)$, show that there exists a unique $T^* \in \mathcal{L}(V,U)$ such that for all $u \in U$, $v \in V$, $\langle Tu, v\rangle_V = \langle u, T^*v\rangle_U$.
   (b) Let $T \in \mathcal{L}(U,V)$. Show that $\ker(T^*) = \operatorname{im}(T)^\perp$, and $\operatorname{im}(T^*) = \ker(T)^\perp$.
   (c) If $T \in \mathcal{L}(U,V)$ is injective, show that $T^*T \in \mathcal{L}(U,U)$ is invertible. Is $TT^* \in \mathcal{L}(V,V)$ invertible?
   (d) Let $W \subseteq \mathbb{R}^n$ be a subspace, and $\{w_1, \dots, w_m\} \subseteq W$ be a basis of W. Let $A \in \operatorname{Mat}(n,m,\mathbb{R})$ be the matrix with $w_1, \dots, w_m$ as columns. Show that $P = A(A^{\mathrm{t}}A)^{-1}A^{\mathrm{t}}$ is the orthogonal projection of $\mathbb{R}^n$ onto W. [You should also justify why $A^{\mathrm{t}}A$ is invertible. Note that given a basis of a subspace, this problem gives you an explicit formula for the orthogonal projection. However, computationally, this formula is about as much 'work' as using Gram-Schmidt first, and using the formula from class.]

4. Suppose $\dim(V) < \infty$ is an inner product space over F, and $T \in \mathcal{L}(V,V)$. Suppose further there exists an orthonormal basis of V consisting of eigenvectors of T.
   (a) If $F = \mathbb{C}$, show that $T^*T = TT^*$.
   (b) If $F = \mathbb{R}$, show that $T^* = T$.

5. Let $F = \mathbb{R}$, and V be an inner-product space. Recall $T \in \mathcal{L}(V,V)$ is orthogonal if $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all $u, v \in V$.
   (a) Show that T is orthogonal if and only if $\|Tu\| = \|u\|$ for all $u \in V$.
   (b) Show that T is orthogonal if and only if $T^* = T^{-1}$.
   (c) Let $V = \mathbb{R}^2$, $T \in \mathcal{L}(V,V)$ be orthogonal, and $B = \{e_1, e_2\}$ be the standard basis. Show that there exists $\theta \in [0, 2\pi)$ such that $\mathcal{M}_B(T) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ or $\mathcal{M}_B(T) = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$. [Elementary trigonometry now shows that the first matrix above is a rotation through angle θ, and the second is a reflection about line $y = \tan(\frac{\theta}{2})x$. Thus all orthogonal transformations on $\mathbb{R}^2$ are either rotations or reflections. The same is true in higher dimensions (and you know enough to prove this).]

6. Let $A \in \operatorname{Mat}(n,n,\mathbb{C})$. We define $e^A = \sum_{k=0}^\infty \frac{1}{k!}A^k$. Show that $\det(e^A) = e^{\operatorname{tr}(A)}$. [Note: An elementary application of the comparison test will quickly show that the series $\sum_{k=0}^\infty \frac{1}{k!}A^k$ converges (absolutely) for any matrix A. For this question, you needn't worry about issues of convergence. The hint is as follows: First check this for diagonal matrices. Next compute what $e^{P^{-1}AP}$ is in terms of P and e^A. Now if your matrix A was diagonalizable, you should have a quick proof. The proof of the general case will follow using the same

idea and question 3 from the previous homework. As a general remark – e^A is called the *matrix exponential*, and is extremely useful in solving ODE's. Proving this question using the 'expansion by minors' formula for the determinant is doomed to failure, but the decomposition of V into (generalized) eigenspaces quickly gives the desired result.]