Homework Assignment 10

Assigned Fri 04/08. Due Fri 04/22.

This homework is a 'double' homework in order to avoid a conflict with Spring carnival. It will count as two homeworks for your grade. **The second half of your homework will be about inner product spaces, etc. and will be added after class on Wed 04/13.**

For the first three questions, we always assume $F = \mathbb{C}$, $\dim(V) < \infty$, $T \in \mathcal{L}(V, V)$. Now we know that the minimal polynomial of T will always factor as a product of (not necessarily distinct) linear factors. We group identical factors together, and subsequently assume that the minimal polynomial of T is $(x - \lambda_1)^{\alpha_1} \cdots (x - \lambda_k)^{\alpha_k}$, where $\lambda_1, \dots, \lambda_k \in F$ are distinct, and $\alpha_1, \dots, \alpha_k \in \mathbb{N}$. Let $E_i' = \ker(T - \lambda_i I)^{\alpha_i}$. [The assumption $F = \mathbb{C}$ can be avoided, but requires a little Field theory to push the proofs through.]

1. (a) Show that for any i, j, $(T - \lambda_i I)^{\alpha_i} \in \mathcal{L}(E_j', E_j')$. Further, if $i \neq j$, then show $(T - \lambda_i I)^{\alpha_i}$ is *invertible* as an element of $\mathcal{L}(E_j', E_j')$.
   (b) Let $p_i = \prod_{j \neq i} (x - \lambda_j)^{\alpha_j}$, and $P_i = p_i(T)$. Show $P_i \in \mathcal{L}(E_i', E_i')$, and is invertible as an element of $\mathcal{L}(E_i', E_i')$.
   (c) Let B_i be a basis of E_i', and $B = \cup_i B_i$. Show that B is linearly independent. [Hint: Apply P_i to a linear relation, and use the previous subpart.]
   (d) Let $p_i = \prod_{j \neq i} (x - \lambda_j)^{\alpha_j}$. Show that there exist $q_1, \dots, q_k \in P_F(x)$ such that $\sum p_i q_i = 1$. [Hint: HW7, Q5(a) might be relevant here.]
   (e) Show that $\operatorname{span}(B) = V$. [Hint: Let $Q_i = q_i(T)$. Show that $P_i Q_i \in \mathcal{L}(V, E_i')$, and $\sum_i P_i Q_i = I$.]

2. The *characteristic polynomial* of T is defined to be $\prod_{i=1}^k (x - \lambda_i)^{m_i}$, where $m_i = \dim(E_i')$. [Usually $m_i = \dim(E_i')$ is called the *algebraic multiplicity* of the eigenvalue λ_i. The *geometric multiplicity* is defined to be $\dim(E_i) = \dim(\ker(T - \lambda_i I))$.]
   (a) Let $S \in \mathcal{L}(U, U)$. Show that $\ker(S^{n+1}) \supseteq \ker(S^n)$. Further, if $\ker(S^{n+1}) = \ker(S^n)$, then for any $m \geqslant n$, $\ker(S^m) = \ker(S^n)$. Conclude that if $S^{n-1} \neq 0$, *and* $S^n = 0$ then $n \leqslant \dim(U)$.
   (b) Show that $\alpha_i \leqslant \dim(E_i')$. [Hint: Let $U = E_i'$, and $S = T - \lambda_i I$. Hopefully the previous subpart helps.]
   (c) *(Cayley-Hamilton theorem.)* If g is the characteristic polynomial of T, show that $g(T) = 0$. [Note that $\deg(g) = \dim(V)$, and consequently the degree of the minimal polynomial of T is at most the dimension of V, as claimed in last weeks homework.]

3. The *determinant* of T is defined to be $\prod_{i=1}^k \lambda_i^{m_i}$. The *trace* of T is defined to be $\sum_{i=1}^k m_i \lambda_i$.
   (a) If $V = \mathbb{C}^2$, and $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, show that the determinant of T is $ad - bc$ and the trace is $a + d$.

   There are explicit formulae for the determinant, and trace in terms of a matrix representation of T. The explicit formula for a determinant is a little complicated involving expansion by minors. In practice, however, this formula is almost never used since it is too computationally intensive. The explicit formula for the trace however is very simple; it's just the sum of the diagonal entries.

   (b) Say $A = (a_{i,j}) \in \operatorname{Mat}(n, n, F)$. Define $\operatorname{tr}(A) = \sum_{i=1}^n a_{i,i}$. If $A, B \in \operatorname{Mat}(n, n, F)$ show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Conclude that if $P \in \operatorname{Mat}(n, n, F)$ is invertible, then $\operatorname{tr}(P^{-1} A P) = \operatorname{tr}(A)$.
   (c) Let C be any basis of V. Show that $\operatorname{tr}(\mathcal{M}_C(T)) = \sum_{i=1}^k m_i \lambda_i$.

The following questions were added Wed 04/13.

4. Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$.
   (a) Show that $|\|u\| - \|v\|| \leqslant \|u - v\|$ for all $u, v \in V$.
   (b) (Unrelated) If for some $u, v \in V$, $|\langle u, v \rangle| = \|u\| \|v\|$, then show that $\exists \alpha \in F$ such that $u = \alpha v$ or $v = \alpha u$.

5. Let V be the vector space of all continuous functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 f(x)^2\,dx < \infty$. For $f, g \in V$, define $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

   (a) Show that $\langle\cdot,\cdot\rangle$ is an inner product on V. If we dropped the requirement that all elements in V are *continuous*, is $\langle\cdot,\cdot\rangle$ still an inner product?

   (b) Let $\mathcal{U} = \operatorname{span}\{1, x, x^2\}$. That is, U is the set of all functions g such that $g(x) = a + bx + cx^2$ for all $x \in [0,1]$. Let P be the orthogonal projection of V onto U, and let $h(x) = \sqrt{1+x}$. Find Ph.

   Recall, from Calculus the Taylor series of $\sqrt{1+x}$ is $1 + \frac{x}{2} - \frac{x^2}{4} + \cdots$. Let's compare how well the second order Taylor polynomial $p_2(x) = 1 + \frac{x}{2} - \frac{x^2}{4}$ and the quadratic polynomial Ph from the previous subpart approximate the function $\sqrt{1+x}$.

   (c) Using a computer, compute $\max\{|h(x) - Ph(x)| \mid x \in [0,1]\}$, and $\max\{|h(x) - p_2(x)| \mid x \in [0,1]\}$. Also plot a graphs of , $|h(x) - Ph(x)|$, $|h(x) - p_2(x)|$.

6. Let V be an inner product space, and $U \subseteq V$ be a non-zero, proper subspace. Let P be the orthogonal projection of U on to V. Find all the eigenvalues of P. Find also the associated eigenspaces. [Recall if λ is an eigenvalue of P, then $E_\lambda = \ker(P - \lambda I)$ is the associated eigenspace.]

7. *(Hard)* Let S be a finite set with n elements. Let C be a collection of subsets of S such that every element of C has an odd number of elements, and any two (distinct) elements of C intersect in an even number of elements. How large can C be? [Clearly C can have at least n elements: Indeed let $C = \{\{s\} \mid s \in S\}$ (i.e. C is the collection of all one element subsets of S). Here C has n elements, and certainly satisfies our requirements above. The question is can you do better? I.e. is there a larger collection C' with the above properties? If yes, how large can it be? If no, prove it.]