21341 Linear Algebra: Final.

Thu 05/05, 2011

- *This is a closed book test. No calculators or computational aids are allowed.*
- *You have 3 hours. The exam has a total of 9 questions and 90 points.*
- *You may use without proof any result that has been proved in class or on the homework, provided you* ***CLEARLY*** *state the result you are using.*
- *For computational questions, I strongly recommend you* ***show all your work****. This way, a computational error leading to the wrong answer will receive partial credit for correct steps.*
- *The questions are roughly in what I perceive as increasing difficulty. They are not in the order material was covered. Good luck.*

In this exam, we always assume V is a vector space over a field F.

$\boxed{10}$ 1. Let $A = \left(\begin{smallmatrix} 3 & 4 \\ 0 & 0 \end{smallmatrix}\right)$. Compute the eigenvalues and singular values of A. [Of course, if A is normal, then the singular values are exactly the absolute value of the eigenvalues of A. But it's not true in general as you should see from this question.]

$\boxed{10}$ 2. Let V be a vector space, and $U \subseteq V$ be a subspace. Suppose $\{u_1, \dots, u_k\} \subseteq U$ is linearly independent and $u_{k+1} \notin U$ then must $\{u_1, \dots, u_{k+1}\}$ be linearly independent? Prove, or provide a counter example.

$\boxed{10}$ 3. State if the following are true or false (no justification is required). A correct answer will get full credit, a blank answer will get half credit, and an incorrect answer will get no credit.

(a) Any $n+1$ distinct vectors in an n dimensional vector space are necessarily spanning.

(b) Let A, B be two linearly independent sets. True or false: $A \cup B$ is necessarily linearly independent?

(c) Let $f, g \in P_F(x)$ be two polynomials with $g \neq 0$. We know there exist $q, r \in P_F(x)$ such that $f = qg + r$ and $\deg(r) < \deg(g)$. True or false: For a given f and g, the q and r with the above properties are unique?

(d) Let $F = \mathbb{C}$, $\dim(V) < \infty$, and $T \in \mathcal{L}(V, V)$ be such that $(T - 3I)(T - 4I)(T - I) = 0$. True or false: T is necessarily diagonalizable.

(e) Let V be an inner product space, $U \subseteq V$ a subspace, and $P \in \mathcal{L}(V, U)$ be the orthogonal projection of V onto U. True or false: P is diagonalizable?

$\boxed{10}$ 4. Let $F = \mathbb{R}$. Let $U = \operatorname{span}\{\left(\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \\ -1 \end{smallmatrix}\right)\}$, and $V = \operatorname{span}\{\left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 2 \\ 1 \end{smallmatrix}\right)\}$ be two subspaces of $\mathbb{R}^3$. Find a basis of $U \cap V$.

$\boxed{10}$ 5. Let V be a finite dimensional inner product space, $U \subseteq V$ be a subspace, and $P \in \mathcal{L}(V, U)$ be the orthogonal projection of V onto U. Show that $P^* = P$.

$\boxed{10}$ 6. Let $F = \mathbb{R}$. Does there exist $A \in \operatorname{Mat}(4, 3, \mathbb{R})$ and $B \in \operatorname{Mat}(3, 4, \mathbb{R})$ such that $AB = I$, where I is the 4×4 identity matrix. If yes, find A and B. If no, prove they don't exist.

$\boxed{10}$ 7. Let V be a finite dimensional vector space over a field F. Suppose for some $\lambda \in F$

$$\{0\} \subsetneq \ker(T - \lambda I) \subsetneq \ker(T - \lambda I)^2$$

Show that there exist $v_1, v_2 \in V$ such that

$$Tv_1 = \lambda v_1 \quad \text{and} \quad Tv_2 = \lambda v_2 + v_1.$$

[Recall $A \subsetneq B$ means that A is a proper subset of B. Also, despite the similarity to your evil 'double' homework #10, this problem can be done directly and independent of the homework. In fact, if you resort to using HW #10, you're probably on the wrong track...]

$\boxed{10}$ 8. Let V be a finite dimensional inner product space. Suppose $T \in \mathcal{L}(V, V)$ is invertible. Show that there exists $c > 0$ such that for all $v \in V$, $\|Tv\| \geqslant c\|v\|$.

$\boxed{10}$ 9. Let V be a finite dimensional vector space over $\mathbb{C}$. Suppose $S, T \in \mathcal{L}(V, V)$ are such that $ST = TS$. Let $\lambda \in \mathbb{C}$ be an eigenvalue of S. Show that there exists $\mu \in \mathbb{C}$ and $v \in V$ with $v \neq 0$, such that *both*

$$Sv = \lambda v \qquad \text{and} \qquad Tv = \mu v$$