Math 341 - Linear algebra

Midterm 2 (A)

Wed. Oct. 27, 2010

1. (2/50) Name:

2. (10/50) Suppose that V is vector space with (real) inner product $\langle \, , \, \rangle$. Prove that for all $u, v \in V$

$$\langle u, v \rangle = \frac{1}{4}[\|u+v\|^2 - \|u-v\|^2].$$

3. Suppose that $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is such that

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix};T\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Let

$$B_1 = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\},$$

and B_2, B' be the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

(a) (3/50) Find $\mathcal{M}(T, B_1, B')$.

(b) (4/50) Find $S_{B_1 \to B_2}$ and $S_{B_2 \to B_1}$.

(c) (3/50) Write down a formula for $\mathcal{M}(T, B_2, B')$ in terms of $\mathcal{M}(T, B_1, B')$ and $S_{B_2 \to B_1}$. You do not have to calculate explicitely $\mathcal{M}(T, B_1, B')$.

4. (10/50) Define $T \in \mathcal{L}(P_2(\mathbb{R}), \mathbb{R}^2)$ by

$$T(p) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix}$$

(where p' denotes the derivative of p). Suppose that we define the following inner product in $P_2(\mathbb{R})$,

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\, dx.$$

Show that with this inner product, $p(x) = x$ is perpendicular to all the polynomials in $N(T)$.

5. Determine whether the following statements are true and false. No justification is required. Each correct answer is worth two points, each blank answer is worth one point, and each incorrect answer is worth zero points.

   (a) (2/50) If A is an $n \times n$ matrix with rank n, then A is invertible.

   (b) (2/50) If $V = S(v_1, \ldots, v_n)$ and $T \in \mathcal{L}(V, W)$, then $W = S(Tv_1, \ldots, Tv_n)$.

   (c) (2/50) If V is a finite dimensional vector space with inner product and B is an orthonormal set in V, then there exists an orthonormal basis of V that contains B. **Hint:** We know that there is a basis of V that contains B.

   (d) (2/50) If V is a vector space with (real) inner product $\langle\ ,\ \rangle$ and $u, v \in V$ are such that

$$\langle u, w \rangle = \langle v, w \rangle$$

   for all $w \in V$, then $u = v$. **Hint:** If $\langle u, w \rangle = \langle v, w \rangle$, then $\langle u - v, w \rangle = 0$.

6. (10/50) Suppose that V is a finite dimensional vector space and $T \in \mathcal{L}(V, V)$ satisfies $R(T) = R(T^2)$. Prove that

$$V = R(T) \oplus N(T).$$

Hint: Show first that $N(T) = N(T^2)$ and use this to prove that $N(T) \cap R(T) = \{0\}$. Conclude then that $R(T) + N(T) = R(T) \oplus N(T)$ coincides with V.