## Math 341 - Linear algebra

Midterm 1 (A)

Wed. Sept. 22, 2010

1. (2/50) Name:

2. (10/50) Suppose that  $\gamma \in \mathbb{R}$  is such that  $\gamma^2 \in \mathbb{Q}$ . Let

$$F := \{a + b\gamma : a, b \in \mathbb{Q}\}.$$

Prove that if  $x \in F$  and  $x \neq 0$  then  $\frac{1}{x} \in F$ . Note: The rest of properties of a field are easy to verify, and hence F is a subfield of the real numbers usually denoted by  $\mathbb{Q}[\gamma]$ . *Hint:* If you carefully write down your steps, in order to avoid expressions with 0 in the denominator, you have to consider two cases, (i)  $\gamma \in \mathbb{Q}$ , (ii)  $\gamma \notin \mathbb{Q}$ .

3. Let

$$U = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in F^2 : x^2 = y^2 \right\}.$$

(a) (5/50) If  $F = \mathbb{R}$ , is U a subspace of  $\mathbb{R}^2$  over  $\mathbb{R}$ ? Justify your answer.

(b) (5/50) If  $F = \{0, 1\}$  with addition and multiplication modulo 2, is U a subspace of  $F^2$  over F? Justify your answer. *Hint:* In this case you can actually list the vectors in U.

- 4. Determine whether the following statements are true and false. Justify your answer. *Hint:* No calculations are needed. Also, in both questions the field of scalars is  $\mathbb{R}$ 
  - (a) (5/50) {1 + x,  $x^2 4$ , 1 + 2x +  $x^2$ , 1} is a linearly independent subset of  $P_2(\mathbb{R})$  (the vector space of polynomials of degree less than or equal to 2 with coefficients in  $\mathbb{R}$ ).

(b)  $(5/50) \{ \binom{1}{2}, \binom{2}{3} \}$  generates  $\mathbb{R}^2$ .

- 5. Determine whether the following statements are true and false. Each correct answer is worth two points, each blank answer is worth one point, and each incorrect answer is worth zero points.
  - (a) (2/50) Let V be a vector space of dimension 7. Suppose that S and T are subspaces of V of dimension 4 and dim  $(S \cap T) = 1$ . Then S + T = V.
  - (b) (2/50) Let  $S_1, S_2, T$  be subspaces of V. If  $S_1 + T = S_2 + T$  then  $S_1 = S_2$ .
  - (c) (2/50)  $S := \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f(0) = f(1)\}$  is a subspace of  $\mathcal{F}(\mathbb{R}, \mathbb{R})$  over  $\mathbb{R}$ . Recall that  $\mathcal{F}(\mathbb{R}, \mathbb{R})$  is the vector space of real-valued functions over  $\mathbb{R}$ .
  - (d) (2/50) If V is a vector space of dimension n (over F, an arbitrary field) and  $\{u_1, u_2, \ldots, u_m\}$  is a set of m > n distinct vectors in V then  $V = S(u_1, \ldots, u_m)$ .

6. (10/50) For  $\alpha \in \mathbb{R}$ , let  $\mathbb{Q}[\alpha]$  be the vector space over  $\mathbb{Q}$  defined by

 $\mathbb{Q}[\alpha] := \{ p(\alpha) : p \text{ is a polynomial with coefficients in } \mathbb{Q} \}.$ 

Suppose that  $\mathbb{Q}[\alpha]$  is finitely generated over  $\mathbb{Q}$  and the dimension of  $\mathbb{Q}[\alpha]$  over  $\mathbb{Q}$  is  $n \in \mathbb{Z}^+$ . Prove that there exists a nonzero polynomial  $p(x) \in P(\mathbb{Q})$  (the vector space of polynomials with coefficients in  $\mathbb{Q}$ ) with degree less than or equal to n such that  $p(\alpha) = 0$ .