Math 341 - Linear algebra

FINAL EXAM

Monday Dec. 13, 2010

1. (2/100) Name:

2. (10/100) Suppose that V is a vector space with **complex** inner product $\langle\ ,\ \rangle$. Prove that for all $v, w \in V$

$$\langle v, w\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|v + i^k w\|^2,$$

where i is the imaginary constant ($i^2 = -1$).

3. Suppose that $V = \mathbb{R}^3$. Consider V as a real inner product space with the dot product in $\mathbb{R}^3$ and let $W = S\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$.

   (a) (15/100) Let T be the orthogonal projection onto W. Find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of T. *Hint:* To answer this question it is not necessary to consider the characteristic polynomial of the matrix of the transformation.

(b) (5/100) Let T be as in (a). Evaluate $T\left(\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right)$

(c) (5/100) Evaluate $D(\mathcal{M}(T, B, B))$, with T as before and B the standard basis of $\mathbb{R}^3$. *Hint:* No calculations are needed to answer this question.

4. Define $T \in \mathcal{L}(P_1(\mathbb{R}), \mathbb{R})$ by

$$T(p) = p(0) + p'(0)$$

(where p' denotes the derivative of p). Suppose that we define the following real inner product in $P_1(\mathbb{R})$,

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

(a) (10/50) Find a polynomial $h \in P_1(\mathbb{R})$ such that $T(p) = \langle p, h \rangle$ for all $p \in P_1(\mathbb{R})$. *Hint:* It is enough to have that $T(1) = \langle 1, h \rangle$ and $T(x) = \langle x, h \rangle$.

(b) (3/50) Prove that the answer that you found in part (a) is unique.

5. Let $\theta \in [0, 2\pi)$ be arbitrary an consider the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) (8/100) Find the eigenvalues of A. *Hint:* Recall that $\cos^2\theta + \sin^2\theta = 1$.

(b) (2/100) As a matrix with complex entries, is A diagonalizable? Justify your answer.

6. Determine whether the following statements are true and false. No justification is required. Each correct answer is worth five points, each blank answer is 3 points, and each incorrect answer is worth zero points.

   (a) (5/100) If V is a finite dimensional inner product space, $v \in V$, $v \neq 0$ and

   $$W = \{w \in V : \langle w, v \rangle = 0\}.$$

   Then

   $$\dim W = \dim V - 1.$$

   .

   (b) (5/100) If V is an inner product space, U is a subspace of V and T is the orthogonal projection onto U then $(Id_V - T)$ is the orthogonal projection onto $U^\perp$.

   (c) (5/100) If A is an $n \times n$ matrix with real entries such that $A^n = I_n$ (the identity matrix) then $D(A) = 1$.

   (d) (5/100) If V is a finite dimensional inner product space and U is invariant under $T \in \mathcal{L}(V, V)$, then $U^\perp$ is invariant under T.

7. (10/100) Prove that if $T \in \mathcal{L}(V, V)$ is a normal operator (i.e. $T^*T = TT^*$) then $N(T) = N(T^*)$. *Hint:* Show first that $\|Tx\| = \|T^*x\|$ for all $x \in V$.

8. (10/100) Suppose that V is a finite dimensional complex vector space, $T \in \mathcal{L}(V, V)$ and 0 is the only eigenvalue of T. Prove that if $n = \dim V$ then $T^n \equiv 0$. *Hint:* Use the Triangular Form Theorem.