Math 341 Final.

Tue, Dec 15, 2009

Time: 50 mins
Total: 80 points

This is a closed book test, and your are not allowed to use calculators or other computational aids. You may use any result from class/homework provided you make an appropriate reference, unless you have been explicitly forbidden from using it. Good luck ⌣̈

Part 1: Computations.

The first few questions in this exam are computational. You are *not* required to show any work, or provide any justification for your answers to this part of the exam. Incorrect answers in this part with no work shown are worth nothing. If you show your work, you might get some partial credit for incorrect answers.

$\boxed{10}$ 1. Find the determinant of the matrix $\begin{pmatrix} 11 & 31 & 31 \\ 0 & 31 & 32 \\ 11 & 31 & 32 \end{pmatrix}$.

2. Let $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ be the linear transformation given by $T\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1+x_2 \\ x_1-x_2 \end{smallmatrix}\right)$. Let $v_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and $v_2 = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and let B be the (ordered) basis (v_1, v_2). [Recall $\mathcal{L}(U, V)$ is the vector spaces consisting of all linear transformations $T : U \to V$. Also an ordered basis is simply an ordered collection of vectors which forms a basis.]

$\boxed{5}$ (a) Express the vectors Tv_1 and Tv_2 in coordinates with respect to the (ordered) basis B.

$\boxed{5}$ (b) Find the matrix of T with respect to the (ordered) basis B.

$\boxed{10}$ 3. Let $\theta \in \mathbb{R}$, and $A = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$. Find the eigenvalues of A (in terms of θ). Find also an orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of the matrix A. [Recall: $\cos^2\theta + \sin^2\theta = 1$.]

Part 2: True or false.

$\boxed{10}$ 4. State if the following are true or false (no justification is required). A correct answer will get full credit, a blank answer will get half credit, and an incorrect answer will get no credit.

(a) The set $\{0, 1, 2, 3, 4, 5\}$ with addition and multiplication defined modulo 6 is a field.

(b) Let V be a vector space, and $S \subseteq V$ be a subset such that whenever $T \subsetneq S$, $\operatorname{span}(T) \subsetneq \operatorname{span}(S)$. Then S is linearly independent. [Recall the notation $A \subsetneq B$ means that A is a proper subset of B. That is $A \subsetneq B$, iff both $A \subseteq B$ and $A \neq B$.]

(c) If $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$, then $\dim(\ker(T)) \geqslant 2$.

(d) If A is an $n \times n$ matrix such that $A^t = -A$ then $\det(A) = 0$.

(e) Let V be a finite dimensional vector space over a field F. Then $T \in \mathcal{L}(V, V)$ is invertible if and only if 0 is not an eigenvalue of T.

Part 3: Short answers.

The next few questions are short answer questions. You are not required to provide any proof or justification for these questions. However, if your short answer is incorrect, but you seem to have some work leading in the correct direction, you may get some partial credit.

$\boxed{5}$ 5. Let F be a field, and U, V be two *finite dimensional* vector spaces over F. Express $\dim(\mathcal{L}(U, V))$ in terms of $\dim(U)$ and $\dim(V)$.

$\boxed{5}$ 6. Spock and Yoda play a game of *determinant tic-tac-toe.* The rules are as follows: We start with an empty 3×3 grid, and Spock and Yoda take turns filling it. Spock goes first. In his turn, a player may put either a 0 or a 1 in any unoccupied grid square. When all 9 squares are filled, we treat this as a 3×3 matrix. If the determinant of this matrix is even Yoda wins. Otherwise, Spock wins.

After 6 turns, the matrix is as follows:

$$\begin{pmatrix} & 0 & \\ & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

It is Spock's turn to play. Can Spock make a move that would guarantee his victory? If yes, what is it?

Part 4: Proofs

You must provide rigorous proofs for your answers to all questions in this part of the exam.

$\boxed{10}$ 7. Prove your answer to question 4(b). Explicitly, let V be a vector space over some field F. Let $S \subseteq V$ be such that for any $T \subsetneq S$, $\operatorname{span}(T) \subsetneq \operatorname{span}(S)$. Decide if S must necessarily be linearly independent or not. If yes, prove it. If no, provide a counter-example.

8. Let A be an $n \times n$ matrix of *real* numbers such that $A^{\mathrm{t}} = -A$.

$\boxed{5}$ (a) Show that there exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of A. [Hint: You may use without proof any theorem proved in class.]

$\boxed{5}$ (b) If $\lambda \in \mathbb{C}$ is an eigenvalue of A, show that $\exists b \in \mathbb{R}$ such that $\lambda = ib$.

Part 5: Real men don't need hints.

The next question is a little 'trickier' than the others. It is possible to get an 'A' on this exam without doing the next question, so I'd recommend attempting this question only if you're convinced you've done all the previous questions to the best of your ability.

$\boxed{10}$ 9. Let p be a prime number, F be the field $\{0, 1, \ldots, p-1\}$ with addition and multiplication defined modulo p. For any $n \in \mathbb{N}$ how many elements of $\mathcal{L}(F^n, F^n)$ are invertible? Justify.