Math 341 Syllabus and Lecture schedule.

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L1, Mon. 8/24.
- Introduction & motivation
- Fields
  - Definition.
  - Uniqueness of inverses. Inverse of inverses.

L2, Wed. 8/26.
  - Multiplication by 0. $(-a)(-b) = ab$, etc.
  - Examples. $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Q}(\sqrt{2})$, etc. (Also some non-examples like $\mathbb{Z}$).

L3, Fri. 8/28.
- Vector spaces
  - Definition.
  - Examples: $0, \mathbb{R}^n, F^n$, function spaces.

L4, Mon. 8/31.
  - Remark that *'head-to-toe'* addition is the same as coordinate addition in $\mathbb{R}^2$.
  - Subspaces. Definition and examples. (Some examples of subspaces in function spaces).
  - Define $\text{span}\{v_1, \ldots, v_n\}$.

L5, Wed. 9/2.
    * Check $\text{span}\{v_1, \ldots, v_n\}$ is a subspace.
    * $U \subseteq V$ a subspace & $u_1, \ldots u_n \in U$ then $\text{span}\{u_1, \ldots, u_n\} \subseteq U$.
    * $\text{span}\{v_1, \ldots, v_n\}$ is the smallest subspace containing $v_1, \ldots, v_n$.
    * Span of infinite sets.

L6, Fri. 9/4.
  - Linear dependence / independence.
    * Example in $\mathbb{R}^3$.
    * $\{u, v\}$ L.D. $\iff u = \lambda v$ or $v = \lambda u$.
  - Basis.
    * Definition, Examples. Canonical basis in F^n.

L7, Wed. 9/9.
    * Any $n + 1$ vectors in $\text{span}\{v_1, \ldots, v_n\}$ are L.D.
    * Any two (finite) basis of V have the same cardinality, and define dimension.

L8, Fri. 9/9.
    * Basis of the zero vector space.
    * S is L.I. $\iff \forall s \in S,\ s \notin \text{span}(S - \{s\})$.
    * If $u \in \text{span}(S)$, then $\text{span}(S \cap \{u\}) = \text{span}(S)$.

L9, Mon. 9/14.
    * Any spanning set contains a basis.
    * Any L.I. set can be extended to a basis.
    * Dimension counting (e.g. $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$.)
    * maximal L.I. $\iff$ minimal spanning $\iff$ basis (On homework)

L10, Wed. 9/16.
    * Row reduction, and explicitly finding basis from spanning sets.

L11, Fri. 9/18.
  - Linear equations.
    * Reduced echelon form.

L12, Mon 9/21.
    * Row rank, dimension of the solution space.
    * The inhomogeneous case (on homework).

L13, Fri 9/25.
- Linear Transformations.
  - Definitions, examples.
  - $\mathcal{L}(U, V)$ is a vector space.

L14, Mon 9/28.
  - Closure of $\mathcal{L}(U, U)$ under composition
  - Associativity, non-commutativity, etc.
  - Identity, inverse.

L15, Wed 9/30.
  - $T : U \to V$ linear and bijective $\implies T^{-1}$ is linear.
  - Isomorphisms
  - $T \in \mathcal{L}(U, V)$ injective $\iff \ker(T) = \{0\}$.

L16, Fri 10/2.
  - $\text{Im}(T)$ is a subspace

- Rank Nullity theorem.

L17, Mon 10/5.
- Coordinates with respect to arbitrary basis.
- Matrix representation of linear transformation.
- $(ST)_B = (S)_B(T)_B$

L18, Wed 10/7.
- Basis change matrix, and relation of columns to the resp. basis vectors.
- Basis change for linear transformations. $(T)_C = S_{C\to B}^{-1}(T)_B S_{C\to B}$

L19, Fri 10/9.
- Inverses of matrices by row reduction.

L20, Mon 10/12.
- Elementary matrices, and row operations through products.

- Inner products
  - Definition, examples.
  - Cosine rule $\iff \langle x, y\rangle = \|x\|\|y\|\cos\alpha$ in $\mathbb{R}^2$.

L21, Wed 10/14.
- Inner product on $\mathbb{R}^n$.
- Lengths. Cauchy-Schwartz, triangle inequality.

L22, Mon 10/19.
- Orthonormal basis. Linear independence and coordinates.

L23, Wed 10/21.
- Gram-Schmidt.
- Orthogonal transformations.

L24, Fri 10/23.
- $T \in L(V, V)$ orthogonal $\iff \|x\| = \|Tx\|$ for all $x \in V$.
- An $n \times n$ matrix is orthogonal $\iff$ the columns form an Orthonormal basis $\iff$ rows form an orthonormal basis $\iff A^{\mathrm{t}}A = I$.

L25, Mon 10/26.
- $T \in L(V, V)$ orthogonal $\iff$ there exists an orthonormal basis $\{u_1, \dots, u_n\}$ such that $\{Tu_1, \dots, Tu_n\}$ is an orthonormal basis $\iff$ there exists an orthonormal basis $B = \{u_1, \dots, u_n\}$ such that $(T)_B^{\mathrm{t}}(T)_B = I$.
- Rotation matrices in $\mathbb{R}^2$.

L26, Fri 10/30.
- Reflection matrices in $\mathbb{R}^2$, and classification of all orthogonal 2×2 matrices.

- Determinants
  - Definition of the determinant function $D : (F^n)^n \to F$.

L27, Mon 11/2.
- Swaps: $D(v_1, \dots, v_{i-1}, v_j, v_{i+1}, \dots, v_{j-1}, v_i, v_{j+1}, \dots, v_n) = -D(v_1, \dots, v_n)$ for $i \neq j$.
- $D(v_1, \dots, v_{i-1}, v_i + \sum_{j\neq i} \alpha_j v_j, v_{i+1}, \dots v_n) = D(v_1, \dots, v_n)$.
- $v_1, \dots, v_n$ linearly dependent implies $D(v_1, \dots, v_n) = 0$.
- Multi-linearity: $D(v_1, \dots, v_{i-1}, v_i' + v_i'', v_{i+1}, \dots, v_n) =$ $D(v_1, \dots, v_{i-1}, v_i', v_{i+1}, \dots, v_n) + D(v_1, \dots, v_{i-1}, v_i'', v_{i+1}, \dots, v_n)$

L28, Wed 11/4.
- Computing by column operations. If $v_1, \dots, v_n$ are L.I., then reduce $v_1, \dots, v_n$ to $e_1, \dots, e_n$ using elementary column operations. Then $D(v_1, \dots, v_n) = (-1)^{\#\text{ swaps}}$/product of all factors you scale by.
- Uniqueness: D, D' two functions satisfying the axioms then $D = D'$.

L29, Fri 11/6.
- Existence: Define v' to be the vector v with first coordinate deleted. Let $v_j = \sum_i a_{ij} e_i$, and define (inductively)

$$D(v_1, \dots, v_n) = \sum_{j=1}^{n} (-1)^{1+j} a_{1,j} D(v_1', \dots, v_{j-1}', v_{j+1}', \dots, v_n')$$

  Prove D satisfies axioms as follows:
  * Show $D(e_1, \dots, e_n) = 1$.
  * Show $D(\alpha v_1, v_2, \dots, v_n) = \alpha D(v_1, \dots, v_n)$.
  * Show $D(v_1 + v_2, v_2, \dots, v_n) = \alpha D(v_1, \dots, v_n)$.
  * Show swaps: $D(v_1, \dots, v_j, \dots, v_i, \dots, v_n) = -D(v_1, \dots, v_n)$ for $i \neq j$.

L30, Mon 11/9.
- $A = (a_{i,j})$. $C_{i,j} = A$ with i^{th} row and j^{th} column deleted. Show $|A| = \sum_j (-1)^{i+j} a_{i,j} |C_{i,j}|$ for any i.
- Show properties of $|A|$ with respect to *row* operations.

L31, Wed 11/11.
- Show $|A| = \sum_i (-1)^{i+j} a_{i,j} |C_{i,j}|$ for any j.
- $|A| = 0 \iff A$ is not invertible.

- Show $|AB| = |A||B|$.

L32, Fri 11/13.

- Eigenvalues
  - $\lambda \in F$ is an eigenvalue of T if $\exists v \neq 0 \ni Tv = \lambda v$. This v is called an eigenvector, with eigenvalue λ. $E_\lambda = \{v \in V \mid Tv = \lambda v\}$ is called the eigenspace of the eigenvalue λ.
  - Characteristic polynomial $f(\lambda) = \det(A - \lambda I)$.
  - Roots of f are exactly eigenvalues of A.
  - If λ is an eigenvalue of A, then $E_\lambda = \ker(A - \lambda I)$, and can explicitly find this by row reduction. (Simple example)
  - If A is 2×2, then $f(\lambda) = \lambda^2 - \operatorname{tr}(A)\lambda + \det(A)$, where $\operatorname{tr}(A)$ is the *trace* of the matrix A, defined to be the sum of the diagonal entries.

L33, Mon 11/16.

  - T is diagonalisable if there exists a basis of V consisting of eigenvectors of T (called an eigenbasis).
  - Matrix of T with respect to the eigenbasis is a diagonal matrix with the eigenvalues on the diagonal.
  - If A is a matrix, $\{v_1, \dots, v_n\}$ an eigenbasis, then $P^{-1}AP = D$ where P is the matrix with $v_1 \dots, v_n$ as columns, D the diagonal matrix with the eigenvalues on the diagonal.
  - Computing A^m for diagonalisable matrices. Computing Fibonacci numbers as an application.

L34, Wed 11/18.

  - *(Unrigorous)* For any (real) matrix A and almost any $v \in \mathbb{R}^n$, $A^n v$ aligns with the eigenspace corresponding to the eigenvalue of A with largest absolute value.
  - Ranking of sports teams in a tournament: Form a matrix A with outcomes of the tournament (e.g. put in the i, j^{th} entry the score of i^{th} team vs the j^{th} team). Such a matrix will necessarily have a positive eigenvalue, and an eigenvector with all positive coordinates. The coordinates of this eigenvector will be the relative ranks of each team.

L35, Fri 11/20.

  - A field F is algebraically closed if every non-constant polynomial with coefficients in F has a root in F. (E.g. $\mathbb{R}$ is not algebraically closed, but $\mathbb{C}$ is).
  - If α is a root of f, then $f(x) = (x - \alpha)^k g(x)$ for some $k \in \mathbb{N}$, and a *polynomial* g such that $g(\alpha) \neq 0$. The number k is called the *multiplicity* of the root α.
  - If $T \in L(V, V)$, pick a basis B of V, and let A be the matrix of T with respect to the basis B. Define the characteristic polynomial of T by $f(\lambda) = \det(A - \lambda I)$.
  - Proof that the definition above is independent of the basis B.
  - If λ is an eigenvalue of T, define the *algebraic multiplicity* of λ to be the multiplicity of λ as a root of the Characteristic polynomial of T. Define the *geometry multiplicity* to be $\dim(E_\lambda) = \dim(\ker(T - \lambda I))$.
  - For example $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ has eigenvalue 1 with both algebraic and geometric multiplicity 1. And $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ has eigenvalue 1 with algebraic multiplicity 2, and geometric multiplicity 1.

L36, Mon 11/23.

  - *(Proof on homework)* The geometric multiplicity is always less than or equal to the algebraic multiplicity.
  - *(Proof on homework)* Let F be algebraically closed. T is diagonalisable if and only if for every eigenvalue of T, the algebraic multiplicity is equal to the geometric multiplicity.
  - Let λ be an eigenvalue of T. We say $v \in V$, $v \neq 0$ is a generalized eigenvector of T with eigenvalue λ if for some $k \geqslant 1$, $(T - \lambda I)^k v = 0$. If $k = 1$, then v is exactly an eigenvector of T.

- While you can not always guarantee the existence of a basis consisting of eigenvectors, you *can* guarantee the existence of a basis consisting of *generalized* eigenvectors. *(This is a consequence of the Cayley-Hamilton theorem, and will be proved in homework.)*
- *(Cayley-Hamilton Theorem)* If V is finite dimensional, F is algebraically closed, $T \in L(V, V)$ and f is the characteristic polynomial of T, then $f(T) = 0$. [The assumption that F is algebraically closed is redundant.]
  * The proof is by induction.
  * If F is algebraically closed, T has an eigenvalue (because the characteristic polynomial has at least one root). Let λ_1 be the eigenvalue, and v_1 be the associated eigenvector.
  * Add vectors $v_2, \dots, v_n$ to get a basis of V, and let A be the matrix of T with respect to this basis. Then A has the form

$$\left(\begin{array}{c|ccc} \lambda_1 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right)$$

  * Thus $f(\lambda) = (\lambda_1 - \lambda)g(\lambda)$, where g is the characteristic polynomial of B.
  * By block multiplication

$$f(A) = (\lambda_1 I - \lambda)g(A) = \left(\begin{array}{c|ccc} 0 & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array}\right) \left(\begin{array}{c|ccc} g(\lambda_1) & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & g(B) & \\ 0 & & & \end{array}\right)$$

  * Since $g(B) = 0$ by the inductive hypothesis, the above product is 0. QED.

L37, Mon 11/30.

- Spectral theorem
  - Let V be a vector space over $\mathbb{C}$. We say $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ be a complex inner product if
    * *(Positive definite)* $\forall v \in V$, $\langle v, v \rangle \in \mathbb{R}$ and $\langle v, v \rangle \geqslant 0$. *(Non-degenerate)* Further $\langle v, v \rangle = 0 \iff v = 0$.
    * *('Bilinear')* $\forall u, v, w \in V$, $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$, and $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$. Further, $\forall \lambda \in \mathbb{C}$, and $u, v \in V$, we have $\langle \lambda u, v \rangle = \bar{\lambda} \langle u, v \rangle$, and $\langle u, \lambda v \rangle = \lambda \langle u, v \rangle$. *Note the* ***complex conjugate*** *when λ is in the first coordinate.*
    * *('Symmetric')* $\forall u, v \in V$, $\langle u, v \rangle = \overline{\langle v, u \rangle}$.
  - All theorems for real inner products have appropriate analogues for complex inner products. For example, the Cauchy-Schwartz inequality: $|\langle u, v \rangle| \leqslant \|u\| \|v\|$.
    * Here's a proof: Case I: Assume $\langle u, v \rangle \in \mathbb{R}$. As before, let $f(\lambda) = \langle u + \lambda v, u + \lambda v \rangle$ for $\lambda \in \mathbb{R}$. This is a quadratic function in λ which is always non-negative, and hence must have a non-positive discriminant. This gives $\langle u, v \rangle \leqslant \|u\| \|v\|$ if $\langle u, v \rangle \in \mathbb{R}$.
    * Case II: $\langle u, v \rangle \in \mathbb{C}$ and $\langle u, v \rangle \neq 0$. Pick $\alpha = \frac{\langle u, v \rangle}{|\langle u, v \rangle|}$. Then $\langle \alpha u, v \rangle = \bar{\alpha} \langle u, v \rangle = |u| v \in \mathbb{R}$. Thus by Case I, $|\langle u, v \rangle| = \langle \alpha u, v \rangle \leqslant \|\alpha u\| \|v\| = |\alpha| \|u\| \|v\| = \|u\| \|v\|$, since $|\alpha| = 1$.
    * Case III: $\langle u, v \rangle = 0$ – immediate.
  - Let $T \in \mathcal{L}(V, V)$. We say $T^* \in \mathcal{L}(V, V)$ is the adjoint of T if $\forall u, v \in V$, $\langle Tu, v \rangle = \langle u, T^* v \rangle$.

- If the adjoint exists, it is unique. That is if T_1, T_2 are two adjoint's of T, then $T_1 = T_2$.
- If V is finite dimensional, then every $T \in \mathcal{L}(V,V)$ has an adjoint.
  * Proof: Case I – $V = \mathbb{C}^n$. Then if A is the matrix of T, immediately check that the linear transformation given by the matrix $A^* = \bar{A}^{\mathrm{t}}$ is the adjoint of T.
  * Case II: Repeat Gram-Schmidt for complex inner-products and prove that V as an orthonormal basis. Now using an orthonormal basis, you can use the formula from the previous case.
- $T \in L(V,V)$ is Hermitian if $T^* = T$.
- If T is Hermitian and λ is an eigenvalue of T then $\lambda \in \mathbb{R}$. (Proof: If $Tv = \lambda v$, then $\bar{\lambda}\langle v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \lambda\langle v, v\rangle$.)
- If T is Hermitian and λ_1, λ_2 are two *distinct* eigenvalues are orthogonal. Proof: Let $Tv_1 = \lambda_1 v_1$, $Tv_2 = \lambda_2 v_2$. Then

$$\lambda_1 \langle v_1, v_2, =\rangle \, \bar{\lambda}_1 \langle v_1, v_2, =\rangle \, \langle Tv_1, v_2\rangle = \langle v_1, Tv_2\rangle = \lambda_2 \langle v_1, v_2\rangle$$

L38, Fri 12/4.

- *(Spectral Theorem)* If $TT^* = T^*T$, then T is diagonalisable by an orthonormal basis. (The converse is also true as long as you insist the eigenbasis is orthonormal).
- If $ST = TS$, then S and T have a common eigenvector.
  * Proof: Let λ be an eigenvalue of T, and $E_\lambda(T) = \ker(T - \lambda I)$. If $v \in E_\lambda(T)$, then $(T - \lambda I)Sv = S(T - \lambda I)v = 0$, and hence $Sv \in E_\lambda(T)$. This shows $S \in \mathcal{L}(E_\lambda(T), E_\lambda(T))$. Since $\mathbb{C}$ is algebraically closed, $S : E_\lambda(T) \to E_\lambda(T)$ must have an eigenvector $v \in E_\lambda(T)$. This is the common eigenvector.
- Proof of the spectral theorem (Forward direction).
  * By induction on $\dim(V)$. If $\dim(V) = 1$ there is nothing to do.
  * Assume the theorem for all vector spaces of dimension $n - 1$. Use the lemma to pick v_1 that is a common eigenvector of T and T^*.
  * Let $W = \{w \mid \langle w, v_1\rangle = 0\}$.
  * Note $T \in \mathcal{L}(W, W)$, since $\langle Tw, v_1\rangle = \langle w, T^* v_1\rangle = \lambda_1^* \langle w, v_1\rangle = 0$.
  * Similarly $T^* \in \mathcal{L}(W, W)$.
  * Let $T|_W$ denote the linear transformation T restricted to the subspace W. Then $(T|_W)^* = T^*|_W$, and hence $(T|_W)^* T|_W = T|_W (T|_W)^*$, and by the inductive hypothesis, there exists $v_2, \ldots, v_n$ an orthonormal basis of W consisting of eigenvectors of $T|_W$.
  * $\{v_1, \ldots, v_n\}$ is the desired basis.