Assignment 13: Assigned Wed 11/25. Due Fri 12/04. *Last ever*

1. The following statements were stated, but not completely proved in class. Prove them.
   - $\star$ (a) If $v_1, \dots, v_n$ are eigenvectors of T corresponding to n *distinct* eigenvalues $\lambda_1, \dots, \lambda_n$, show that $v_1, \dots, v_n$ are linearly independent. [This is a special case of the next subpart, so you can ignore it if you can do the next subpart without trouble. If you have trouble with the next subpart, it might help you to try this one first. A hint would be to use induction, assume $\sum c_i v_i = 0$, and apply $T - \lambda_1 I$ to both sides of this expression.]
   - (b) Let F be algebraically closed, V be a finite dimensional vector space over F, $T \in \mathcal{L}(V, V)$, and $\lambda_1, \dots, \lambda_k$ be k distinct eigenvalues of T. Let $E_{\lambda_1}, \dots, E_{\lambda_k}$ be the associated eigenspaces, and for each $i \in \{1, \dots, k\}$, let $\{v_{i,1}, v_{i,2}, \dots, v_{i,\dim(E_{\lambda_i})}\}$ be a basis of E_{λ_i}. Show that
   $$\bigcup_{i=1^k} \bigcup_{j=1}^{\dim(E_{\lambda_i})} \{v_{i,j}\}$$
   is linearly independent.
   - (c) With V, F, T as above, show that T is diagonalisable *if and only if*, for every eigenvalue λ, the geometric multiplicity of λ equals the algebraic multiplicity of λ.
2. If A is an $n \times n$ matrix over an algebraically closed field and $\det(A) \neq 0$, then show that there exists a polynomial g of degree $n-1$ such that $g(A) = A^{-1}$.
3. Here's an alternate proof of the Cayley-Hamilton theorem. (Consequently, you may not use the Cayley-Hamilton theorem to prove anything in this subpart.)
   - (a) Show that the Cayley-Hamilton theorem is true for $n \times n$ upper triangular matrices. [Recall a matrix $A = (a_{i,j})$ is *upper triangular* if $a_{i,j} = 0$ whenever $i > j$.]
   - (b) If F is algebraically closed, V is finite dimensional and $T \in \mathcal{L}(V, V)$, then show that there exists a basis B such that the matrix of T with respect to B is an upper triangular matrix. Now use the previous subpart to prove the Cayley-Hamilton theorem for T. [Hint: Use induction on the dimension of V.]

Since this is the last 341 homework, I leave you with an important Theorem. While you can't always find a basis of eigenvectors, you can always find a basis of generalized eigenvectors. This problem outlines a proof.

4. As usual, let F be an algebraically closed field, V a finite dimensional vector space over F and $T \in \mathcal{L}(V, V)$. Let $\lambda_1, \dots \lambda_k$ be the (distinct) eigenvalues of T with *algebraic* multiplicities $m_1, \dots, m_k$ respectively. Let $E'_i = \ker(T - \lambda_i I)^{m_i}$.
   - (a) For any i, show that $T \in \mathcal{L}(E'_i, E'_i)$. Hence show that $\forall i, j$, $(T - \lambda_i I)^{m_i} \in \mathcal{L}(E'_j, E'_j)$.
   - (b) Show that $\dim(E'_i) = m_i$. [Hint: Pick a basis of E_i, and extend it to a basis of V. Now see if you can write the matrix of T in this basis in 'block' form, and use it to your advantage. The Cayley-Hamilton theorem helps.]
   - (c) If $i \neq j$ show that $(T - \lambda_i I)^{m_i}$ is *invertible* as an element of $\mathcal{L}(E'_j, E'_j)$.
   - (d) Let $\{v_{i,1}, v_{i,2}, \dots, v_{i,m_i}\}$ be a basis of E'_i. Show that $\{v_{i,j} \mid i \leqslant k, j \leqslant m_i\}$ is linearly independent.
   - (e) Show that $\{v_{i,j} \mid i \leqslant k, j \leqslant m_i\}$ above is a basis of V. What does the matrix of T with respect to this basis look like?