Assignment 12: Assigned Wed 11/18. Due Tue 11/24 by 10:00AM

Please note the due date is *Tuesday at 10:00AM* because there will be no class the Wednesday before Thanksgiving. You may turn in your homework in class on Monday 11/23, or slip it under my door anytime before 10:00AM on Tuesday 11/24.

1. (a) Diagonalise the matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. That is, find a matrix P such that $P^{-1}AP$ is a diagonal matrix. Use this to compute an explicit formula for A^n.

   (b) Recall, the Fibbonacci numbers are defined by $f_0 = 0$, $f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. In class we saw that $\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = A^n \begin{pmatrix} f_1 \\ f_0 \end{pmatrix}$, where A is the matrix from the previous subpart. Use the previous subpart to compute an explicit formula for f_n.

   (c) Suppose we define a different 'Fibbonacci' sequence by $f_{n+1} = 4f_n - 5f_{n-1} + 2f_{n-2}$. Show that for some appropriate matrix A,

   $$\begin{pmatrix} f_{n+2} \\ f_{n+1} \\ f_n \end{pmatrix} = A^n \begin{pmatrix} f_2 \\ f_1 \\ f_0 \end{pmatrix}$$

   Diagonalize this matrix as above, and use this to find a formula for f_n explicitly in terms of f_0, f_1 and f_2.

2. If A is an $n \times n$ matrix, we define it's *trace* (denoted by $\operatorname{tr}(A)$) to be the sum of the diagonal entries. Explicitly, if $A = (a_{i,j})$, $\operatorname{tr}(A) = \sum_{i=1}^n a_{i,i}$.

   (a) If A, B are $n \times n$ matrices, show that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Consequently if $B = P^{-1}AP$ for some $n \times n$, invertible matrix P, show that $\operatorname{tr}(B) = \operatorname{tr}(A)$.

   (b) If V is a n dimensional vector space, and $T \in L(V, V)$, we define $\operatorname{tr}(T)$ as follows: Pick *any* basis B of V. Let A be the matrix representation of T with respect to the basis B. Define $\operatorname{tr}(T) = \operatorname{tr}(A)$. Show that this is *well defined*. Namely, show that $\operatorname{tr}(T)$ is independent of the basis B.

3. If A is an $n \times n$ matrix, show that it's charactoristic polynomial is of the form

   $$f(\lambda) = (-\lambda)^n + \operatorname{tr}(A)(-\lambda)^{n-1} + \cdots + \det(A)$$

   [This generalizes what we saw in class for 2×2 matrices.]

4. Let F be algebraically closed, V be a finite dimensional vector space over F, $T \in L(V, V)$.

   (a) If λ is an eigenvalue of T, show that the geometric multiplicity of λ is less than or equal to the algebraic multiplicitly of λ.

   (b) If T is diagonalisable, then show that for every eigenvalue λ, the algebraic and geometric multiciplicities are equal.