Assignment 11: Assigned Wed 11/11. Due Wed 11/18

1. In class we showed that if A is an invertible $n \times n$ matrix, then $|AB| = |A||B|$ for any $n \times n$ matrix B. This problem proves $|AB| = |A||B|$ if A is not invertible.

   If A, B are two $n \times n$ matrices, and A is not invertible, show that AB is not invertible, and hence conclude $|AB| = |A||B|$.
2. **Section 18.** 2, 6.
3. Let $A = (a_{i,j})$ be an $n \times n$ matrix, and for every i, j let $C_{i,j}$ be the matrix A with the i^{th} row and j^{th} column deleted. Recall, we defined $|A|$ inductively by the formula $\sum_j (-1)^{1+j} a_{1,j} |C_{1,j}|$.
   (a) *(Easy)* Using only this formula, show that if two columns of A are identical, then $|A| = 0$.
   (b) *(A little harder)* Using only this formula, show that if two rows of A are identical, then $|A| = 0$.

   NOTE: Both the results in the above questions follow *immediately* using the properties of determinants proved in class. What we did in class was to show that that the formula above defines a determinant function (on either the rows or the columns as vectors), and then these properties follow immediately from the definition of determinant. What I'm asking for in this problem is a *direct* proof of this fact. That is, using only the above formula, and ignoring everything that was done in class/the book, prove the above two subparts.
4. *Cramer's rule.* With notation from the previous problem, we define $A^\dagger$ to be a matrix who's entry in the i^{th} row and j^{th} column is $(-1)^{i+j}|C_{j,i}|$. *Note, this is $C_{j,i}$ and not $C_{i,j}$.* Show that $AA^\dagger = |A|I$ (the right hand side is the determinant of A times the identity matrix). This gives the well known, explicit, and practically useless formula for the inverse of a matrix $A^{-1} = \frac{1}{|A|}A^\dagger$.

   [HINT: Write out the i, j^{th} entry in the matrix $AA^\dagger$ as a sum. With a little manipulation of indexes, you should find that this sum is exactly the determinant of some matrix. Compute the determinant of this matrix for $i = j$, and for $i \neq j$ to finish this problem.]
5. One can recursively expand out the formula for the determinant, and get an explicit (non-recursive) formula for the determinant in terms of the matrix entries. This problem explains how to do this.

   Let $n \in \mathbb{N}$, and F be a field. We say $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is a *permutation* if σ is bijective. We define the *signature* of a permutation, denoted by $\varepsilon(\sigma)$ by

   $$\varepsilon(\sigma) = D(e_{\sigma(1)}, e_{\sigma(2)}, \dots, e_{\sigma(n)})$$

   Here $e_1, \dots, e_n$ are the standard basis vectors of F^n, and $D : (F^n)^n \to F$ is the determinant.
   (a) Show that for any permutation σ, $\varepsilon(\sigma) = \pm 1$.
   (b) If σ, τ are two permutations then notice that $\sigma \circ \tau$ is also a permutation. Show that $\varepsilon(\sigma \circ \tau) = \varepsilon(\sigma)\varepsilon(\tau)$.
   (c) If $A = (a_{i,j})$ is an $n \times n$ matrix, then show that

   $$|A| = \sum_{\sigma} \varepsilon(\sigma) a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

   where the sum above is taken over all permutations σ.

   [Note that this formula has $n!$ terms in it, each with a coefficient of ± 1. Note further that each term in this sum involves exactly one element from each row of A, and exactly one element from each column of A. Another way of saying this is that pick any n entries of the matrix A, such that you pick exactly one entry from each row, and exactly one entry from each column. If we add all such terms together *choosing the correct sign* then we get the determinant of A.]