Assignment 9: Assigned Wed 10/21. Due *Wed 11/04*

1. **Section 15.** 1(b) & (c), 4, 6

⋆ 2. *(A little harder)* Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is such that for all $x, y \in \mathbb{R}^n$, $\|T(x) - T(y)\| = \|x - y\|$, and $T(0) = 0$. Show that T is linear.

⋆ 3. *(Challenge, that will earn a small reward.)* Let S be a finite set with n elements. Let C be a collection of subsets of S such that every element of C has an odd number of elements, and any two (distinct) elements of C intersect in an even number of elements. How large can C be? [Clearly C can have at least n elements: Indeed let $C = \{\{s\} \mid s \in S\}$ (i.e. C is the collection of all one element subsets of S). Here C has n elements, and certainly satisfies our requirements above. The question is can you do better? I.e. is there a larger collection C' with the above properties? If yes, how large can it be? If no, prove it.]

The following questions were added Wed 10/28

4. **Section 15.** 8, 9, 12
5. *(Follow up to 9)* This question defines 'orthogonal projections', and introduces basic properties of it. (It's not as 'hard' as some of the other 'last questions' on other homework, since this week you have an optional problem and challenge to keep you busy if you so choose.)

   Let V be a vector space over $\mathbb{R}$, with inner product $\langle \cdot, \cdot \rangle$, and let $W \subseteq V$ be a subspace.

   (a) If V is finite dimensional, show that for any $v \in V$, there exists a pair of vectors w, w' such that $w \in W$, $w' \in W^\perp$ and $v = w + w'$.

   For the remaining subparts, you can not assume V is finite dimensional. You may however assume that for any $v \in V$, there exists $w \in W$, $w' \in W^\perp$ such that $v = w + w'$. (This can be proved in the infinite dimensional setting, provided we make one additional assumption on V. Wikipedia 'Hilbert space' if you're curious.)

   (b) Show that the pair of vectors w, and w' satisfying the conditions in the previous subpart are unique.

   (c) For any $v \in V$, we define $Pv = w$, where w is the (unique) vector from the previous subpart. The operator P is called the *orthogonal projection onto* W. Show that P is linear, and $\|Pv\| \leqslant \|v\|$.

   (d) *(Generalization of 13)* If $v = w_1 + w_2$ where $w_1 \in W$, then show that $\|w_2\| \geqslant \|v - Pv\|$. [I claim that geometrically, this means that amongst all vectors connecting a point and a plane, the perpendicular has the shortest length. It would be good to see if you can relate this problem to this geometric intuition. It of course won't help with the proof.]

   (e) Show that for any $u, v \in V$, $\langle Pv, w \rangle = \langle v, Pw \rangle$.

   (f) If $V = \mathbb{R}^n$, and $\langle \cdot, \cdot \rangle$, show that the matrix of P is symmetric (i.e. $P^{\mathrm{t}} = P$). [You can, but need not, do this using only the result from the previous subpart.]

Assignment 10: Assigned Wed 11/04. Due Wed 11/11

1. **Section 16.** 1(c), 2, 3, 4.
2. **Section 17.** 1, 2. [You can do both these without the explicit formula for D.]
3. Let V be an n-dimensional vector space over F. Suppose $D : V^n \to F$ is a function such that

   (a) For any $i \leqslant n$, $j \neq i$, and $v_1, \dots v_n \in V$, we have $D(v_1, \dots, v_{i-1}, v_i + v_j, v_{i+1}, \dots v_n) = D(v_1, \dots, v_n)$.

   (b) For any $\alpha \in F$, $i \leqslant n$, and $v_1, \dots v_n \in V$, we have $D(v_1, \dots, v_{i-1}, \alpha v_i, v_{i+1}, \dots v_n) = \alpha D(v_1, \dots, v_n)$.

   Show that D is unique upto scalar multiplication. Namely, if D, D' are two functions with the above propoerties, then show that there exists $\alpha \in F$ such that either $D = \alpha D'$ or $D' = \alpha D$.
4. Let V be an n-dimensional vector space over F. Let $\mathcal{D}_2$ be the set of all functions $D : V^2 \to F$ which satisfy the following propoerties:

   (a) $\forall v \in V$, $D(v, v) = 0$.

   (b) $\forall u, v, w \in V$, $D(u + v, w) = D(u, w) + D(v, w)$ and $D(u, v + w) = D(u, v) + D(u, w)$.

   (c) $\forall \alpha \in F$, $u, v \in V$, $D(\alpha u, v) = \alpha D(u, v) = D(u, \alpha v)$.

   Show that $\mathcal{D}_2$ is a vector space, and find it's dimension. [Hint: In finding the dimension of $\mathcal{D}_2$, you might want to do it for $V = F^n$ first.]