Assignment 7: Assigned Wed 10/07. Due Wed 10/14

1. **Section 13.** 1, 7, 11
2. **Section 12.** 3(b) & (d), 4, 5
3. Let U be a finite dimensional vector space over a field F, and $T \in \mathcal{L}(U, U)$. Show that the following are equivalent:
   (a) T is invertible.
   (b) T is injective.
   (c) T is surjective.
4. Let $T \in L(U, V)$, $B = (b_1, \dots, b_m)$ an ordered basis of U, and $C = (c_1, \dots, c_n)$ be an ordered basis of V.
   (a) Show that there exists a matrix M such that for all $u \in U$, $(Tu)_C = M(u)_B$. What are the dimensions of this matrix? What meaning/significance do the columns of this matrix have (in relation to the two basis B and C)?
   (b) Show that $\dim(\operatorname{Im}(T)) = \text{col. rank}(M)$. [Here col. rank($M$) denotes the column rank of M]
   (c) Show that $\dim(\ker(T)) = m - \text{row rank}(M)$. [Here row rank($M$) denotes the row rank of M]
   (d) Show that the row rank and column rank of M are equal.

Assignment 8: Assigned Wed 10/14. Due Wed 10/21

1. **Section 12.** 7(c), (e) [For 7, feel free to use any method you like, not just that in Example C].
2. If $S, T \in L(U, U)$ are invertible, show ST is invertible. Express $(ST)^{-1}$ in terms of S^{-1} and T^{-1}.
3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find a necessary and sufficient condition for A to be invertible. Find an explicit formula for A^{-1}.
4. Let A be a $m \times n$ matrix. We define the transpose of the matrix A^{t} (sometimes denoted by A^*) to be the matrix obtained by interchanging the rows and columns of the matrix. For example, $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}^{\mathrm{t}} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$. If A is an $m \times n$ matrix, B is an $n \times n'$ matrix, show that $(AB)^{\mathrm{t}} = B^{\mathrm{t}} A^{\mathrm{t}}$.
5. Let $\{v_1, \dots, v_n\} \subseteq \mathbb{R}^m$, and suppose you want to find a basis for $V = \operatorname{span}\{v_1, \dots, v_n\}$. The algorithm used so far is as follows: Put the vectors in echelon form, and pick the non-zero ones. They will be a basis of V.

   Suppose now you wanted to reduce $\operatorname{span}\{v_1, \dots, v_n\}$ to a basis of V. That is you wanted to find some subset of $\{v_1, \dots, v_n\}$ which is a basis of V. In this case, the above algorithm will not help you! Here's an algorithm that works: First form a matrix with $v_1, \dots, v_n$ as *columns*. Now put this matrix in *row* reduced echelon form. Now, for each i, if the i^{th} *column* in the reduced matrix contains a leading 1, pick the vector v_i to be part of your basis. If the column does not contain a leading 1, don't pick the vector. This algorithm will give you a method to reduce $\{v_1, \dots, v_n\}$ to a basis of V.
   (a) Using the above algorithm, reduce the following sets to a linearly independent set with the same span.
   (i) $\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \right\}$ (ii) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -8 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ -5 \\ -1 \end{pmatrix} \right\}$
   (b) Prove that the above algorithm works in general.
   (c) Explain how you would adapt this algorithm if $\mathbb{R}^m$ above was replaced with an abstract (finite dimensional) vector space U.
6. Let $V = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous, and } \int_{-\infty}^{\infty} f(x)^2 \, dx < \infty \}$.
   (a) Show that V is a vector space. [There is only one axiom that is hard to check. Only prove this one.]
   (b) Let $\langle f, g \rangle = \int_{-\infty}^{\infty} f(x) g(x) \, dx$. Show that this defines an inner product on V.
   (c) Show that if $f, g \in V$, $|\int_{-\infty}^{\infty} f(x) g(x) \, dx| \leqslant (\int_{-\infty}^{\infty} f(x)^2 \, dx)^{\frac{1}{2}} (\int_{-\infty}^{\infty} g(x)^2 \, dx)^{\frac{1}{2}}$.