Assignment 5: Assigned Fri 09/25. Due Wed 09/30

1. (a) Write down an algorithm that will reduce any given matrix to row reduced echelon form using elementary row operations (i.e. by reordering rows, scaling rows by a non-zero factor, and replacing any particular row by the sum of itself and a scalar multiple of a different row).
   ⋆ (b) Implement the above algorithm with a computer program.
2. Show that any collection of non-zero vectors in echelon form are linearly independent.
3. Find a basis for the following subspaces
   (a) $\operatorname{span}\{\begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\3\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\-1\end{pmatrix}\}$.
   (b) $\operatorname{span}\{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}\}$.
4. Find the null space of the following matrices. Also find a basis for the null space.
   (a) $A = \begin{pmatrix}0 & -1 & 2 & 4\\ 2 & 2 & 1 & 1\end{pmatrix}$
   (b) $A = \begin{pmatrix}1 & 0 & 2 & 1\\ 2 & 0 & 1 & 0\\ -1 & 2 & 0 & 4\end{pmatrix}$.
5. Let A be an m by n matrix, and $b \in \mathbb{R}^m$. Let $c_1, \dots, c_n$ be the columns of A (treated as vectors in $\mathbb{R}^m$). Show that the system of equations $Ax = b$ has a solution (in $\mathbb{R}^n$) if and only if $b \in \operatorname{span}\{c_1, \dots, c_n\}$.
6. With the notation from the previous subpart, let A' be the m by $n+1$ matrix obtained by augmenting the column b to the matrix A (explicitly, A' is the matrix who's columns are the vectors c_1, …, c_n, b). Show that the system $Ax = b$ has a solution (in $\mathbb{R}^n$) if and only if the *row* rank of A' equals the *row* rank of A.

Note: As I mentioned in class, the rows and columns of a matrix represent different quantities. The columns of a matrix form a basis of the image (the content of Question 5). The rows of the matrix are 'orthogonal' to the kernel (related to Question 6). We will address both these issues later in the semester.

Assignment 6: Assigned Wed 09/30. Due Wed 10/07

1. **Section 11.** 1, 4, 6(b) & (d), 7, 8 (only test 6(a)&(c))
2. Suppose U and V are any two n-dimensional vector spaces over a field F. Show that U and V are isomorphic.
3. (a) Given an example of a field F, two (finite dimensional) vector spaces U, V over F, and two linear transformations $S \in \mathcal{L}(U, V)$, $T \in \mathcal{L}(V, U)$ such that $ST = I$, but $TS \neq I$. [Note $ST \in L(V, V)$, so when we say $ST = I$, we really mean ST equals the identity element in $L(V, V)$ (sometimes denoted by I_V, or 1_V). Similarly for $TS = I$, we mean $TS = I_U = 1_U$.]
   (b) Suppose U is a *finite dimensional* vector space over F, and $S, T \in L(U, U)$ are such that $ST = I$. Show that $TS = I$.
   (c) Show that the previous subpart is *false* if U is not finite dimensional. Namely find an infinite dimensional vector space, and two linear transformations $S, T \in L(U, U)$ such that $ST = I$ but $TS \neq I$.
4. (a) Suppose V is a vector space over $\mathbb{R}$, and $T : V \to V$ is such that $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}$. Show that for all $q \in \mathbb{Q}$, and $v \in V$, we have $T(qv) = qT(v)$. [Surprisingly, we need not have $T(\alpha v) = \alpha T(v)$ for all $\alpha \in \mathbb{R}$. The existence of such functions however involves the axiom of choice.]
   (b) Suppose now $V = \mathbb{R}$, and $T : \mathbb{R} \to \mathbb{R}$ satisfies $T(x + y) = T(x) + T(y)$ for all $x, y \in \mathbb{R}$. If further T is continuous, show that $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbb{R}$.