Assignment 3: Assigned Wed 09/09. Due Wed 09/16

1. **Section 4.** 9, 10.
2. **Section 5.** 1, 3, 4, 5.
3. Let p be prime, $F = \{0, 1, \dots, p-1\}$ with addition and multiplication defined modulo p. (This is called 'the finite field of order p'). Let $n \in \mathbb{N}$. How many one dimensional subspaces does F^n have?
4. (a) Let V be a vector space, and I some (possibly infinite) set. Suppose for every $i \in I$, we are given a set C_i, which is a linearly independent subset of V. Suppose further for every $i, j \in I$, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$. Show that $\bigcup_{i \in I} C_i$ is a linearly independent subset of V. [Recall that an infinite set S is called linearly independent if *every* finite subset is linearly independent. This problem is the key step in showing that every (non necessarily finitely generated) vector space has a basis. As soon as you prove the result in this problem, the axiom of choice (or more precisely *'Zorn's Lemma'*) will guarantee the existence of a maximal linearly independent set. From class, we know this must be a basis.]

   (b) The analogue of the above subpart for spanning sets is false! Find a counter example. Namely, let V be a vector space. Find a (possibly infinite) set I and a collection of sets $\{C_i \mid i \in I\}$ such that for every $i \in I$, $C_i \subseteq V$, and $\operatorname{span}(C_i) = V$. Further, for every $i, j \in I$, either $C_i \subseteq C_j$ or $C_j \subseteq C_i$. However $\operatorname{span}(\bigcap_{i \in I} C_i) \subsetneq V$.

Assignment 4: Assigned Wed 09/16. Due *Fri 09/25*

This homework is intentionally shorter than your other homework assignments in light of your midterm. It will also be accepted on Friday, instead of on doomsd…erm. Wednesday.

1. If S is linearly independent, and $u \notin \operatorname{span}(S)$, then show that $S \cup \{u\}$ is linearly independent. [I used this fact in class, but did not write down an iron clad proof of it. Note also the implicit assumptions in the problem: Namely we assume V is a vector space over some field F, and that $S \subseteq V$. Further we assume that $u \in V$ (or more precisely $u \in V - S$). Once you get accustomed to it, this is what your problems will look like.]
2. Let V be a vector space over some field F. Prove that the following statements are equivalent:
   (a) B is a basis of V.
   (b) B is a maximal linearly independent subset of V. Namely B is linearly independent, and if $C \supsetneq B$, then C is linearly dependent.
   (c) B is a minimal spanning set. Namely $\operatorname{span}(B) = V$, and if $C \subsetneq B$, then $\operatorname{span}(C) \subsetneq V$.

   [The usual way to prove that a list of statements are equivalent is to prove (a) implies (b), (b) implies (c) and finally (c) implies (a). While you're free to use your own favourite set of implications, I recommend following the above structure in this case.]
3. Let W be a finitely generate vector space, and $U, V \subseteq W$. Let $B = \{z_1, \dots, z_k\}$ be a basis of $U \cap V$, with the convention that if $U \cap V = \{0\}$, then $k = 0$ and $B = \emptyset$. Extend B to a basis of U by adding $C = \{u_1, \dots, u_m\}$ (i.e. $B \cup C$ is a basis of U). Again we use the convention that if $U \cap V = U$, then $m = 0$, and $C = \emptyset$. Similarly, extend B to a basis of V by adding $D = \{v_1, \dots, v_n\}$. Show that $B \cup C \cup D$ is a basis of $U + V$. Conclude $\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V)$.