Assignment 2: Assigned Wed 09/02. Due Wed 09/09

Problems marked with a $*$are optional, and should *not* be turned in with your regular homework. Numbered problems refer to problems in your textbook (Curtis)

$\star$ 1. **Section 3.** 1, 2, 3, 4.

2. **Section 3.** 6, 9 [You might want to read pages 21–26 before doing these problems.]
3. **Section 4.** 1, 3, 5, 7, 8.[For 1, 3, just 'Yes' or 'No' will suffice. No need to provide verbose justification.]
4. Let $U \subseteq \mathbb{R}^3$ be the region defined by $x_1^2 + 2x_2^2 \leqslant 3x_3^2$. Draw a sketch of U, and find an English word that describes it's shape. Show that U is not a subspace of $\mathbb{R}^3$.
5. The conclusion of both subparts below will follow directly from a general theorem we will prove later. However it's worth while doing them out explicitly by hand at least once…
   (a) Suppose u_1, u_2 are *any* two linearly independent vectors in $\mathbb{R}^2$, then show (by direct computation) that $\operatorname{span} u_1, u_2 = \mathbb{R}^2$.
   (b) Let $V = \mathbb{R}^3$, and $U \subseteq \mathbb{R}^3$ be the plane $x_1 + x_2 + x_3 = 0$. Show (by direct computation) that if u_1, u_2 are *any* two linearly independent vectors in U, then $U = \operatorname{span}\{u_1, u_2\}$.
6. Here's another proof showing that the dimension of a vector space is well defined.
   (a) Let F be any field, $m < n \in \mathbb{N}$, and $\alpha_{ij} \in F$ be given. Show that there exists $x_1, x_2, \dots, x_n \in F$ *not all* 0 such that

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \cdots + \alpha_{1n}x_n &= 0 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \cdots + \alpha_{2n}x_n &= 0 \\ &\vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \cdots + \alpha_{mn}x_n &= 0. \end{aligned}$$

   [Hint: In words this problem says that any system of homogeneous linear equations has a non-zero solution, *provided* you have more variables than equations. The hint is to use induction. But don't get carried away and use some sort of fancy double induction trick on m and n. You can do this directly with induction on one of the variables.]

   (b) Suppose now V is a vector space over F, $m < n \in \mathbb{N}$, and $V = \operatorname{span}\{u_1, \dots, u_m\}$. Show (using the previous subpart) that any subset of n vectors in V must be linearly dependent. [Hint: Let $v_1, \dots, v_n$ be n vectors in V, and express each v_j as a linear combination $\sum_i \alpha_{ij} u_i$. Now somehow reduce linear dependence of v_j's to solving equations like in the previous subpart. Of course, as we've seen in class, this subpart immediately implies that any two (finite) basis in a vector space have the same number of elements.]

   (c) The statements in parts (a) and (b) above are really equivalent. Above you should have shown that part (a) implies part (b). Now do the converse: Namely, assuming the result of part (b) above, show that the result in part (a) is true.