Assignment 14: Assigned Sat 12/05. Due *Never.*

1. (a) Show that $(T^*)^* = T$.
   (b) Show that $(ST)^* = T^* S^*$.
2. Let $S, T \in L(V, V)$ be such that $ST = TS$.
   (a) If S and T are diagonalisable, show that there exists a basis of V consisting of common eigenvectors of S and T.
   (b) However, not every eigenbasis of S is an eigenbasis of T. Give an example of S, T such that $ST = TS$, and an eigenbasis of S which is not an eigenbasis of T.
3. Prove the converse of the spectral theorem: Namely, if $T \in \mathcal{L}(V, V)$ has an orthonormal eigenbasis, then show $TT^* = T^*T$.
4. Let V be a finite dimensional complex vector space with innerproduct $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$. We say $T \in \mathcal{L}(V, V)$ is unitary if $T^* = T^{-1}$ (this is the complex analogue of 'orthogonal').
   (a) Show that T is unitary if and only if for every orthonormal basis $\{v_1, \dots, v_n\}$ of V, the set $\{Tv_1, \dots, Tv_n\}$ is an orthonormal basis of V.
   (b) If λ is an eigenvalue of T, show that $|\lambda| = 1$.
   (c) Is T diagonalisable?
5. Let V be a finite dimensional vector space over $\mathbb{R}$, and $\langle \cdot, \cdot \rangle$ be a (real) innerproduct on V. Let $T \in \mathcal{L}(V, V)$.
   (a) If T is symmetric, show that T is diagonalisable, and further the eigenbasis can be chosen to be orthonormal.
   (b) Conversely, if T has an orthonormal eigenbasis, show T must be symmetric? (Note how much more general the complex case was!).
   (c) *(A little harder, and I won't put up a solution for this part.)* If however $T^*T = TT^*$, show that there exists a basis of V such that the matrix of T with respect to this basis has the form

$$\begin{pmatrix}
\lambda_1 \\
& \ddots \\
& & \lambda_k \\
& & & r_1 \cos\theta_1 & r_1 \sin\theta_1 \\
& & & -r_1 \sin\theta_1 & r_1 \cos\theta_1 \\
& & & & & \ddots \\
& & & & & & r_{k'} \cos\theta_{k'} & r_{k'} \sin\theta_{k'} \\
& & & & & & -r_{k'} \sin\theta_{k'} & r_{k'} \cos\theta_{k'}
\end{pmatrix}$$