Math 341 Homework (Fall 2009)

Assignment 1: Assigned Wed 08/26. Due Wed 09/02

1. Suppose F (with binary operations $+, \cdot$) is some field, and assume all quantities referenced in this problem are elements of F. Do this problem using *only* the basic field axioms.
   (a) Show that the additive (and multiplicative) identity in a field is unique. [To help you get started, here's what you should do: Suppose 0 and $0'$ were two (additive) identities. Show that $0 = 0'$.]
   (b) Show that $-a = (-1)a$. [To explain the notation, the left hand side is the additive inverse of a. The right hand side is the additive inverse of 1 multiplied by a.]
   (c) If $a \neq 0$, and $ab = ac$ then show $b = c$.
   (d) Show that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, provided $b, c \neq 0$. [We *define* $\frac{a}{b} = a \cdot b^{-1}$, where b^{-1} is the (unique) multiplicative inverse of b.]
2. Show that the set $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ forms a field. [Associativity, commutativity and distributivity follow from the analogous properties of real numbers, so don't bother checking them. The only properties you should explicitly check are that sums and products of elements in $\mathbb{Q}(\sqrt{2})$ are in $\mathbb{Q}(\sqrt{2})$, and the existence of additive and multiplicative inverses.]
3. Suppose the set $F = \{0, 1, \alpha, \beta\}$ with binary operations $+$ and $\cdot$ form a field. Construct the addition and multiplication for F (i.e. draw two tables – one showing how to add all possible pairs of elements in F, and the other showing how to multiply all possible pairs of elements in F). [It turns out that there are exactly two ways one can do this. You don't have to justify your steps or work for this question. Just the final answer will suffice.]
4. Let F be a field, and $n \in \mathbb{N}$. We define $n! \in F$ by
$$n! = \underbrace{(1 + 1 + \cdots + 1)}_{n\text{-times}}(n-1)!$$
and $0! = 1$. Notice that the standard definition $n! = n(n-1)!$ can't really be used here, since there is no reason for any given natural number n to be an element of the filed F. However, no matter what the field F is, we are guaranteed $1 \in F$, so our definition makes sense (and further $n! \in F$).

   Now if $n, m \in \mathbb{N}$ with $m \leqslant n$, we define the binomial coefficient $\binom{n}{m}$ by $\binom{n}{m} = \frac{n!}{m!(n-m)!} \in F$. If $a, b \in F$ and $n \in \mathbb{N}$ show that
$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
5. (a) Suppose F is a field with an even number of elements (i.e. the cardinality of the set F is both finite and even). Show that $1 + 1 = 0$ in F. [Hint: Show first that $\exists a \in F$ such that $a \neq 0$ and $a + a = 0$.]
   (b) Suppose now F is a field with an odd number of elements. Show that $1 + 1 \neq 0$ in F.
6. (a) Suppose F is a *finite* set, and that $+, \cdot$ are two binary operations on F which satisfy all the field axioms *except* (possibly) the existence of multiplicative inverses. Suppose further, $\forall a, b \in F$, $ab = 0$ implies that $a = 0$ or $b = 0$. Show that F must in fact be a field.
   (b) Let p be prime, and $F = \{0, 1, \ldots, p-1\}$. Define $+$ and $\cdot$ to be addition and multiplication (respectively) modulo p. (That is, for $a, b \in F$ define $a + b$ to be the remainder of the sum of a and b when divided by p.) Show that F is a field. [The only field axiom you should explicitly check is the existence of multiplicative inverses.]