Math 341 Midterm 2.

Wed, Oct 28, 2009

Time: 50 mins
Total: 60 points

This is a closed book test, and your are not allowed to use calculators or other computational aids. You may use any result from class/homework provided you make an appropriate reference, unless you have been explicitly forbidden from using it. Good luck ◡̈

[10] 1. Let V be a vector space over $\mathbb{R}$, and $\langle\ ,\ \rangle$ an inner-product on V. Suppose $u, v \in V$ are such that $\langle u, v\rangle = 0$, then show that $\|u\|^2 + \|v\|^2 = \|u+v\|^2$. State (without proof) a 'high-school geometry' result that this problem is a generalization of.

[10] 2. Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

[10] 3. Let $x = \left(\begin{smallmatrix} 3 \\ 4 \\ 1 \end{smallmatrix}\right) \in \mathbb{R}^3$. Let $b_1 = \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right)$, $b_2 = \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$, and $b_3 = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, and let $B = (b_1, b_2, b_3)$ be an (ordered) basis of $\mathbb{R}^3$. Express the vector x in coordinates with respect to the (ordered) basis B. [Remember that elements of a set have no defined order. In order to talk about coordinates with respect to a basis, we need the basis to be ordered, which is why we used the notation $B = (\dots)$, instead of $B = \{\dots\}$. If this confuses you, treat $B = \{b_1, b_2, b_3\}$, and ignore the word 'ordered' above.]

4. Let F be a field, and U, V be two vector spaces over F. Let W be a subspace of U. We define $W' \subseteq \mathcal{L}(U, V)$ by

$$W' = \{T \in \mathcal{L}(U, V) \bigm| \forall w \in W,\ T(w) = 0\}$$

[Recall $\mathcal{L}(U, V)$ is the set of all linear transformations from U to V. To write out the above definition in words, W' is the set of all linear transformations $T : U \to V$, such that for every $w \in W$, $T(w) = 0$.]

[5] (a) Show that W' is a subspace $\mathcal{L}(U, V)$.

[5] (b) If both U and V are finite dimensional, express $\dim(W')$ in terms of $\dim(U), \dim(V)$ and $\dim(W)$. No proof or justification is required. Just make an intelligent guess.

[10] 5. Let F be a field, and U, V be two *finite dimensional* vector spaces over F. If $\dim(U) > \dim(V)$, and $T \in \mathcal{L}(U, V)$ then show that T is not be injective. [Recall 'injective' is the same as 'one-to-one'.]

The next question is a little 'trickier' than the others. It is possible to get an 'A' on this exam without doing the next question, so I'd recommend attempting this question only if you're convinced you've done all the previous questions to the best of your ability.

[10] 6. Let V be a finite dimensional vector space over $\mathbb{R}$, and $\langle\ ,\ \rangle$ be an inner product on V. If $T \in \mathcal{L}(V, \mathbb{R})$, show that there exists $u_0 \in V$ such that for all $v \in V$, $T(v) = \langle u_0, v\rangle$. [To remind you: We think of $\mathbb{R}$ as a (one dimensional) vector space over the field $\mathbb{R}$, and $T \in \mathcal{L}(V, \mathbb{R})$ means $T : V \to \mathbb{R}$ is a linear transformation.]