# Math 341 Midterm

Wed, Sep 23, 2009

Time: 50 mins
Total: 60 points

*This is a closed book test, and your are not allowed to use calculators or other computational aids. You may use any result from class/homework provided you make an appropriate reference, unless you have been explicitly forbidden from using it. Good luck* ◡̈

[10] 1. Let $F$ be a field for which $1 + 1 = 0$. True or false: For any $x, y \in F$ must we have $(x + y)^2 = x^2 + y^2$. Prove your answer. [Recall for any $a \in F$, $a^2$ is defined to be $a \cdot a$.]

[10] 2. State whether each of the following statements are true or false. No justification is required. Each correct answer is worth two points, each blank answer is worth one point, and each incorrect answer is worth zero points.

(a) The set $F = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is a field. [Recall $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ is the set of integers.]

(b) Let $V$ be the vector space (over $\mathbb{R}$) consisting of all functions with domain $\mathbb{R}$ and range $\mathbb{R}$ (i.e. $V = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$). Let $U \subseteq V$ be defined by $U = \{f \in V \mid f(0) = 1\}$. Then $U$ a subspace of $V$. [Recall $\mathbb{R}$ is the set of real numbers.]

(c) Let $V$ be as in the previous subpart, and let $W \subseteq V$ be defined by $U = \{f \in V \mid f(1) = 0\}$. Then $W$ a subspace of $V$.

(d) Let $V$ be a vector space over some field $F$. If $S, T \subseteq V$ are two linearly independent sets, then $S \cup T$ is linearly independent.

(e) Let $V$ be the vector space of all real numbers, over the field of rational numbers. The dimension of $V$ is finite.

3. Let $F$ be a field, $V$ be a vector space over $F$, and $\{e_1, e_2, \dots, e_{341}\}$ be a basis of $V$.

[10] (a) Let $T \subseteq V$ be *any* collection of 340 vectors in $V$. Can $\operatorname{span}(T) = V$? Prove your answer.

[10] (b) Let $S = \{e_i + e_j \mid i, j \in \{1, \dots, 341\}\ \&\ i < j\}$. Is $S$ linearly independent? Prove your answer.

[10] 4. Let $V$ be a vector space over some field $F$. Suppose $U$, $W$ are two subspaces of $V$ such that $U \cap W = \{0\}$. Let $S = \{u_1, \dots, u_m\}$ be a subset of $U$, and $T = \{w_1, \dots, w_n\}$ be a subset of $W$. If $S$ and $T$ are linearly independent, show that $S \cup T$ is also linearly independent. [While you can use induction to provide a lengthy convoluted solution of this problem, there is a short direct solution which does not involve induction.]

The next question is a little 'trickier' than the others. It is possible to get an 'A' on this exam without doing the next question, so I'd recommend attempting this question only if you're convinced you've done all the previous questions to the best of your ability.

[10] 5. Prove your answer to Question 2 (e). Namely, let $V$ be the vector space of all real numbers, over the field of rational numbers. Decide if the dimension of $V$ is finite or not, and prove it.