LINEAR ALGEBRA FINAL: TAKE HOME PART

JAMES CUMMINGS

This part of the final is due by 8:30am on Mon Dec 18. You may work on it during any continuous 24 hour period of your choice. You may not collaborate. You may consult your notes and any other material you wish, and may also ask for a hint from me (one per customer).

You may find it helpful to recall that if p is a polynomial in λ, then $p(a) = 0$ if and only if $\lambda - a$ divides p. In this case the roots of p will be a together with the roots of the quotient polynomial $p/(\lambda - a)$.

(1) Find the two complex solutions of the quadratic equation $z^2 - (4+2i)z + (2+4i) = 0$.

(2) Find the three complex solutions of the cubic equation $z^3 + 8 = 0$.

(3) Consider the 4×4 symmetric matrix

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial is $\lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda$, you need not prove this.

(a) Find the eigenvalues of A.

(b) Find an orthonormal basis (orthonormal with respect to the standard inner product on $\mathbb{R}^4$) consisting of eigenvectors of A.

(4) Let V be a vector space and let S, T both be linear maps from V to V. We will say that v is a *simultaneous eigenvector of S and T* if v is both an eigenvector of S and an eigenvector of T (possibly with different eigenvalues).

Show that if V has a basis consisting of simultaneous eigenvectors of S and T, then $ST = TS$.

(5) Consider the 2×2 hermitian matrix

$$B = \begin{pmatrix} 1 & 1+i \\ 1-i & 1 \end{pmatrix}$$

(a) Find the characteristic equation of B.

(b) Solve it to find the eigenvalues of B.

(c) Find an orthonormal basis (orthonormal with respect to the standard inner product on $\mathbb{C}^2$) consisting of eigenvectors of B.