ON THE HOMOGENIZATION OF DIFFUSIONS IN PERIODIC COMB-LIKE STRUCTURES

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Abstract

In this thesis we study the homogenization of diffusions in two particular comb-like structures. In both models, the comb can be viewed as a macroscopic diffusion with a trapping mechanism. The processes spend non-trivial amounts of time in these traps and convergence is established using martingale problems and excursion theory. The limiting process has an explicit form as time-changed Brownian motion and also as the unique solution to a certain system of SDE. The limiting macroscopic process is also shown to be a trapped diffusion whose Kolmogorov equation has a term with fractional-time derivatives.

Acknowledgments

First I would like to thank my advisors Gautam Iyer and Robert Pego for their time and patience with our discussions these past few years. These meetings certainly have shaped the way I will view mathematics for years to come. I would also like to thank our collaborator Jim Nolen for his critical insights and contributions to this work. Without his input I would likely have been floundering for significantly longer on this project.

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To my non-mathematician friends, thanks for helping me stay grounded as I dove into the world of mathematics. I've found that one can quickly disconnect from reality when you're only surrounded by things that make perfect sense. Finally, a big thank you to my family who have stayed supportive of my work regardless of the setbacks.

Contents

1. **Introduction**

Much of the novel material in this thesis can be found in an upcoming paper jointly written by myself, Gautam Iyer, James Nolen and Robert Pego [CINP18]. In particular Theorem 4.2, Proposition 5.5, Theorem 6.1 and the supporting lemmas are taken from the paper.

1.1. **Homogenization.**

The field of *homogenization* is concerned with models which have parameters that are rapidly oscillating in space. It is impractical to measure all of the parameters for a model which has microscale fluctuations in applications; instead, there is a concentration on obtaining a limiting macroscopic model by averaging the coefficients. A simple example of periodic averaging is the well known result that if $f \in L^2(\mathbb{R})$ is periodic, meaning $f(x+n) = f(x)$ for all $n \in \mathbb{Z}^d$, then

$$f\Big(\frac{x}{\varepsilon}\Big) \rightharpoonup \int_{[0,1]^d} f(x)\,dx \quad \text{weakly as } \varepsilon \to 0\,. \tag{1.1}$$

The most commonly studied sources of the averaging in homogenization of partial differential equations derive from either a periodic or random environment. In the case of linear divergence form periodic elliptic operators, the equations have the form

$$-\nabla \cdot (a^\varepsilon(x)\nabla u^\varepsilon(x)) + b^\varepsilon(x)\nabla u^\varepsilon = f^\varepsilon \quad \text{for } x \in \mathbb{R}^d \tag{1.2}$$

where $a^\varepsilon \geqslant \alpha^\varepsilon I$ for some $\alpha^\varepsilon > 0$ and $a^\varepsilon(x+\varepsilon n) = a^\varepsilon(x)$, $b^\varepsilon(x+\varepsilon n) = b^\varepsilon(x)$ for all $n \in \mathbb{Z}^d$. The period structure leads to averaging in the weak sense.

In the stochastic case, the problem has a random structure which is not periodic per say, but the distribution of the structure is periodic and ergodic with respect to shifts. The averaging then follows from some version of the ergodic theorem (see [BLP78, Koz79] for the original treatment and [AKM17] for a recent book with the quantitative theory).

In the periodic homogenization of elliptic and parabolic equations where the coefficients stay bounded and don't degenerate, the structure of the homogenized model is usually an equation of the same form as the original model but with constant coefficients. The most basic result of this form is the elliptic case where a^ε takes the form $a^\varepsilon(x) = a\Big(\frac{x}{\varepsilon}\Big)$ where a is periodic. In this case, the Dirichlet problem on a domain Ω takes the form:

$$-\nabla \cdot \Big(a\Big(\frac{x}{\varepsilon}\Big)\nabla u^\varepsilon(x)\Big) = f(x) \quad \text{for } x \in \Omega\,, \tag{1.3}$$

$$u^\varepsilon = 0 \quad \text{for } x \in \partial\Omega\,. \tag{1.4}$$

The standard result is that there exists a constant matrix $\bar{a}$ such that $u^\varepsilon \rightharpoonup u$ weakly in H_0^1 where u solves the homogenized problem.

$$\begin{aligned} -\nabla \cdot \Big(\bar{a}\nabla u(x)\Big) &= f(x) \quad && \text{for } x \in \Omega\,, \\ u &= 0 && \text{for } x \in \partial\Omega\,. \end{aligned}$$

One might expect that the homogenized coefficient matrix $\bar{a}$ would be obtained by a simple averaging as in (1.1), but it turns out there is an additional term. The effective diffusion matrix can be written as:

$$\bar{a}(y) = \int_{\mathbb{T}^d} a(y)(I + \nabla\chi(y)^T)\,dy$$

where $\chi : [0,1]^d \to \mathbb{R}^d$ is the "corrector" whose rows χ_i satisfy the cell problems:

$$-\nabla \cdot (a(y)(\nabla \chi_i(y) + e_i)) = 0 \qquad \text{with } \chi_i \text{ periodic.} \tag{1.5}$$

If one reinterprets the above equation as saying $\chi_i(y) + y_i$ is a-harmonic for each i, we see that χ_i is the error between standard coordinates and the a-harmonic coordinates.

In contrast with the above setting, if degeneracy is introduced to the microscopic model, the homogenized model may not be so easily decoupled from the microstructure. In these situations, the limit model is a set of equations, one for the macroscopic limit which is coupled through a boundary condition to an equation in the microscopic domain. As we will see, the model we study in this thesis is of this type (see equations (1.7)-(1.8) below).

Since the conception of the field of homogenization, a diverse set of tools have been developed to establish the convergence of solutions to periodic equations to the solution of the homogenized equation. Originally, practitioners would use formal asymptotic expansions to guess the effective limit equations [KK73]. The idea was to assume u^ε took the form

$$u^\varepsilon(x) = u_0\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon^2 u_2\Big(x, \frac{x}{\varepsilon}\Big) + \dots$$

where $u_i(x,y)$ is periodic in y. Then we plug this ansatz into (1.2) and match orders of ε to get a hierarchy of equations. We can often close the system of equations using the periodicity assumption in the second variable and then lowest order term gives the macroscopic equation for the limit. Once the proposed limiting equation is established, one can actually prove convergence of the solutions using a variety of methods, for example compensated compactness methods used by Murat and Tartar [MT97].

A more direct approach is the two-scale convergence method, developed by Nguetseng, which uses test functions with a periodic variable to establish a type of weak convergence to the function $u_0(x,y)$ in the asymptotic expansion above, without supposing such an expansion is possible outright [Ngu89]. More precisely, $u^\varepsilon(x) \xrightarrow{2} u_0(x,y)$ if for all test functions $\Psi \in \mathbb{C}_c^\infty(\mathbb{R} \times (0,1)^d)$):

$$\int_{\mathbb{R}^d} u^\varepsilon(x)\Psi\Big(x, \frac{x}{\varepsilon}\Big)\, dx = \int_{\mathbb{R}^d}\int_{\mathbb{T}^d} u_0(x,y)\Psi(x,y)\, dy\, dx\,.$$

By integrating against such test functions and using compactness theorems for two-scale convergence, the appropriate effective equation can be computed directly.

The periodic unfolding method developed in [ADH90] and formalized in [CDG02] transforms functions of the macro variable into a function of both the micro and macro variables. The idea is to introduce an unfolding operator $\mathcal{T}^\varepsilon$ defined by

$$\mathcal{T}^\varepsilon(f)(x,y) = f\Big(\varepsilon\Big\lfloor\frac{x}{\varepsilon}\Big\rfloor + \varepsilon y\Big) \qquad \text{for } x \in \mathbb{R}^d, y \in \mathbb{T}^d\,.$$

It can be shown that $u^\varepsilon(x) \xrightarrow{2} u_0(x,y)$ if and only if $\mathcal{T}^\varepsilon(u^\varepsilon)(x,y) \rightharpoonup u_0(x,y)$ weakly in $L^2(\mathbb{R}^d \times [0,1]^d)$, which connects compactness results for two-scale convergence to usual weak compactness in L^2.

For the homogenization of non-linear elliptic PDE, Evans applied the perturbed test function method to study the limits of problems $A^\varepsilon(u^\varepsilon)$ where A^ε is an elliptic operator in non-divergence form which is highly oscillating in a periodic variable [Eva89]. First one shows the solutions have Hölder regularity and so up to a subsequence $u^\varepsilon \to u$ uniformly. The goal is to show u is a viscosity solution to a certain elliptic PDE, $A(u) = 0$ which is independent from the periodic variable. We say u is a subsolution if for any smooth ϕ such that $\phi(x_0) = u(x_0)$ and $\phi \geqslant u$ near x_0, $A(\phi)(x_0) \leqslant 0$. Likewise, u is a supersolution if for any smooth ϕ such that $\phi(x_0) = u(x_0)$ and $\phi > u$ near x_0, $A(\phi)(x_0) \geqslant 0$ and u. A viscosity solution is any function who is both a subsolution and supersolution. If we want to show u is a subsolution, we consider ϕ such that $u - \phi$ has a strict local max and aim to show $A(\phi)(x_0) \leqslant 0$. The idea of the perturbed test function method is that we can create a new test function $\phi^\varepsilon = \phi + \varepsilon^2\psi(x/\varepsilon)$ and then $u^\varepsilon - \phi^\varepsilon$ has a local max at x_ε near x_0. Then, classical comparison principles can be applied to get an inequality satisfied by ϕ^ε at x_ε and convergence of $\phi^\varepsilon \to \phi$ and $x_\varepsilon \to x$ gives the desired inequality for ϕ.

Other methods for homogenization utilize alternative interpretations of elliptic and parabolic equations. Energy methods relate solutions to partial differential equations and the critical points of certain "energy functionals". The idea is for certain classes of elliptic partial differential equations, the solution u^ε to the PDE is the function which minimizes a particular functional $\mathcal{F}^\varepsilon : L^2 \to \mathbb{R}$. Then since we want to show $u^\varepsilon \to u$ where u is the solution to some limiting model with corresponding energy functional $\mathcal{F}$. For this purpose, a type of convergence of the functionals $\{\mathcal{F}^\varepsilon\}_{\varepsilon\to 0}$ was developed by De Giorgi in [DGF75] called Γ-convergence. An important property of Γ-convergence is that if $\mathcal{F}^\varepsilon$ converges to $\mathcal{F}$, then the minimizers also converge. More precisely if u^ε minimizes $\mathcal{F}^\varepsilon$, $\mathcal{F}^\varepsilon \xrightarrow{\Gamma} \mathcal{F}$ and $u^\varepsilon \to u$, then u is a minimizer for $\mathcal{F}$. For a full treatment of the properties of Γ-convergence and on homogenization via Γ-convergence see the book by Dal Maso [DM12].

1.2. **Homogenization of SDE.** The notion of homogenization can also be extended to the setting of Stochastic Differential Equations. The oscillating microstructure can be reinterpreted as a "fast diffusion" which lives on an $O(1/\varepsilon^2)$ time scale. The goal of homogenization is to find a limiting homogenized macroscopic diffusion living on an $O(1)$ time scale through averaging. In this case, the averaging is obtained through ergodicity of the fast process.

Let W be a Brownian motion, $a \in C^1(\mathbb{T}^d; R^{d\times d}_{\text{Sym}})$ uniformly elliptic and choose σ such that $a = \sigma\sigma^T$. Consider the diffusion process which solves

$$dX^0_t = \nabla \cdot a(X^0_t)\, dt + \sigma(X^0_t) dW_t\,,$$
$$X^0_0 = x\,.$$

Then let $\pi : \mathbb{R}^d \to \mathbb{T}^d$ be the projection map and $Y^0_t = \pi(X^0_t)$ be the projection of this process onto $\mathbb{T}^d$. The associated generator takes the form

$$A^0 = \nabla \cdot (a\nabla) = a : \nabla^2 + \nabla \cdot a\nabla\,,$$

which is symmetric with respect to the Lebesgue measure on $\mathbb{T}^d$ and hence the Lebesgue measure is the unique invariant measure for Y^0. Recall the corrector equation (1.5) and define χ_ℓ as the unique bounded periodic solution to

$$A^0\chi_\ell = -\ell \cdot \nabla a\,. \tag{1.6}$$

We now rescale the process to obtain the homogenized limit. Let

$$X^\varepsilon_t = \varepsilon X^0_{t/\varepsilon^2} = \varepsilon \int_0^{t/\varepsilon^2} \nabla \cdot a(X^0_s)\, ds + \varepsilon \int_0^{t/\varepsilon^2} \sigma(X^0_s)\, dW_s\,,$$

and define $Y^\varepsilon_t = \pi(X^\varepsilon_t)$. Using Itô's formula and (1.6) one finds that

$$Y^\varepsilon_t \cdot \ell = \varepsilon \int_0^{t/\varepsilon^2} \sigma^T(Y^0_s)(\ell - \nabla\chi_\ell(Y^0_s)) \cdot dW_s + \varepsilon(\chi_\ell(Y^0_t) - \chi_\ell(Y^0_0)) .$$

As χ_ℓ is bounded, the second term converges uniformly to 0 and so the limiting behavior for Y^ε_t as $\varepsilon \to 0$ is the same as the martingale

$$M^\varepsilon_t = \varepsilon \int_0^{t/\varepsilon^2} \sigma^T(Y^0_s)(\ell - \nabla\chi_\ell(Y^0_s)) \cdot dW_s .$$

If we look at the quadratic variation, we see

$$\langle M^\varepsilon\rangle_t = \varepsilon^2 \int_0^{t/\varepsilon^2} a(Y^0_s)(\ell - \nabla\chi_\ell(Y^0_s)) \cdot (\ell - \nabla\chi_\ell(Y^0_s))\, ds$$

$$\xrightarrow{\varepsilon\to 0} t \int_{\mathbb{T}^d} a(y)(\ell - \nabla\chi_\ell(y)) \cdot (\ell - \nabla\chi_\ell(y))\, dy ,$$

by the ergodic theorem. A proof of the following convergence theorem can be found in [Oll94].

Theorem 1.1. *The rescaled process Y^ε_t converges in law as $\varepsilon \to 0$ to a Brownian motion B_t with diffusion matrix satisfying*

$$\bar{a}(y) = \int_{\mathbb{T}^d} a(y)(I + \nabla\chi(y)^T)\, dy .$$

Recall that this is the same diffusion matrix from §1.1.

1.3. **The Clark Model.** The motivation for this work is a study of the parabolic *double porosity limit* by Clark [Cla98], where the author considers a flow in a medium composed of "blocks", where the permeability is low, and "fissures" where the permeability is relatively high (see Figure 1). In these blocks, the low permeability is modeled with a degenerating diffusion coefficient. Because of the slow diffusion in the blocks, we can think of the blocks as traps in which the diffusion can get stuck. A more precise description of the model is as follows:

Let Ω_B be an open set with Lipschitz boundary such that $\bar{\Omega}_B \subset \mathbb{T}^2$. We denote by B the periodic extension to all of $\mathbb{R}^2$. Let $F = \mathbb{R}^2 \backslash \bar{B}$. The fluid flow in this situation is modeled by the system

$$\partial_t u^\varepsilon - \nabla \cdot \left(a^\varepsilon \nabla u^\varepsilon\right) = f , \tag{1.7}$$

$$u^\varepsilon_0(x) = \mathbf{1}_F\Big(\frac{x}{\varepsilon}\Big)u_0(x) + \mathbf{1}_B\Big(\frac{x}{\varepsilon}\Big)U_0\Big(x, \frac{x}{\varepsilon}\Big) . \tag{1.8}$$

Here $\varepsilon > 0$ is the cell size, $\mathbf{1}_F$ and $\mathbf{1}_B$ are indicator functions of the fissures and blocks respectively. The functions u_0 and U_0 make up the initial fluid density, where $U_0 = U_0(x, y)$ is periodic in the second variable. The diffusivity a^ε is given by

$$a^\varepsilon(x) = \mathbf{1}_F\Big(\frac{x}{\varepsilon}\Big)a\Big(\frac{x}{\varepsilon}\Big) + \varepsilon^2\mathbf{1}_B\Big(\frac{x}{\varepsilon}\Big)A\Big(\frac{x}{\varepsilon}\Big),$$

where a and A are uniformly elliptic matrices.

The distinguishing feature of this model is that the diffusivity in the blocks vanishes like ε^2 as $\varepsilon \to 0$. This is in contrast with (1.3) where the ellipticity is uniform in ε. The main result in [Cla98] shows that as $\varepsilon \to 0$, u^ε converges to the solution of a coupled system in which the fluid in the fissures is driven by a nonlocal boundary integral of the density of the fluid in the blocks, and the fluid in the blocks is coupled to the fluid outside through a boundary condition.

Explicitly, the effective system is

$$\begin{aligned} \partial_t u - \nabla_x \cdot \big(\bar{a} \nabla_x u\big) + Q = f, &\quad \text{for } x \in \Omega,\ t > 0\,,\\ \partial_t U - \nabla_y \cdot \big(A \cdot \nabla_y U\big) = 0\,, &\quad \text{for } y \in \Omega_B,\ t > 0\,,\\ U(x,y,t) = u(x,t), &\quad \text{for } y \in \partial\Omega_B,\ x \in \Omega\,,\\ U(x,y,0) = U_0(x,y), &\quad \text{for } x \in \Omega,\ y \in \Omega_B\,,\\ u(x,0) = u_0(x), &\quad \text{for } x \in \Omega\,. \end{aligned}$$

Here $Q = Q(x,t)$ is defined by

$$Q(x,t) = \int_{y \in \partial\Omega_B} A(y) \nabla_y U(x,y,t)\, d\nu(y)\,,$$

and $\bar{a}$ is a uniformly elliptic matrix representing the effective diffusivity. Clark proved this using the two-scale convergence method mentioned in §1.1. In this thesis, we study a 1D graph analogue of the Clark model, which captures the essential character of the degeneracy in the diffusion rate, and study it using the associated diffusion as described in §1.2.

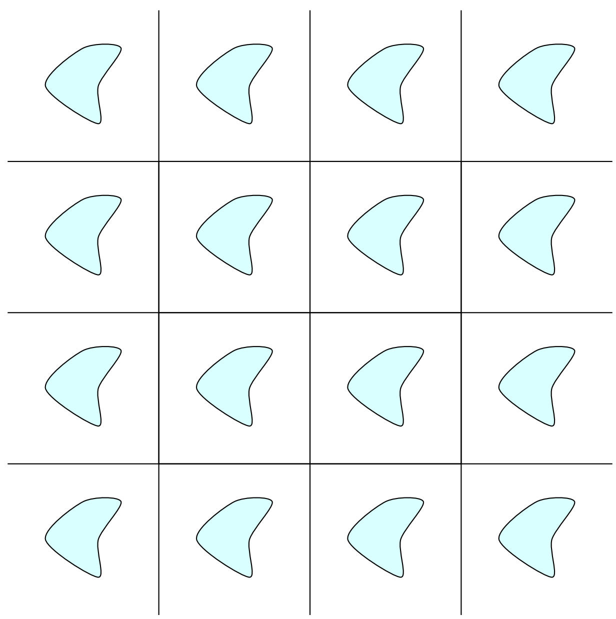

FIGURE 1. A decomposition of $\mathbb{R}^2$ into fissures and blocks.

1.4. **The Model and Main Theorem.** The aim in this thesis is to understand two toy models and their limit behavior from a probabilistic point of view. The first is a one-dimensional version of the Clark model constructed on the infinite comb Ω'_ε defined by

$$\Omega'_\varepsilon \stackrel{\text{def}}{=} \big(\mathbb{R} \times \{0\}\big) \cup \big(\varepsilon\mathbb{Z} \times [0, \varepsilon)\big)\,.$$

We think of this domain as an infinite connected graph embedded in $\mathbb{R}^2$ consisting of a spine $\mathbb{R}\times\{0\}$ along with an infinite collection of teeth $\varepsilon\mathbb{Z}\times[0,\varepsilon)$ glued to the spine at the vertices $\varepsilon\mathbb{Z}\times\{0\}$, which we will call the junction points. We parameterize the comb by (x,y) with $x\in\mathbb{R}$, and $y\in[0,\varepsilon]$, with the constraint that $y=0$ whenever $x\notin\varepsilon\mathbb{Z}$ being understood. The spine $\mathbb{R}\times\{0\}$ represents the fissures where diffusion will be relatively fast, while the teeth represent the blocks where diffusion will be slow. In this scenario, the simplest analogue of (1.7) is

$$\partial_t u^\varepsilon - \frac{1}{2}\partial_x^2 u^\varepsilon = 0 \quad \text{when } x\notin\varepsilon\mathbb{Z},\ y=0\,,$$
$$\partial_t u^\varepsilon - \frac{\varepsilon^2}{2}\partial_y^2 u^\varepsilon = 0 \quad \text{when } x\in\varepsilon\mathbb{Z},\ y\in(0,\varepsilon)\,,$$

along with the flux continuity condition at the junction points:

$$\frac{\varepsilon^2}{2}\partial_y u^\varepsilon + \frac{1}{2}\partial_x^+ u^\varepsilon - \frac{1}{2}\partial_x^- u^\varepsilon = 0 \quad \text{when } x\in\varepsilon\mathbb{Z},\ y=0\,. \tag{1.9}$$

Here ∂_x^- and ∂_x^+ refer to the left and right derivatives respectively. For simplicity, we impose a homogeneous Neumann condition at the free end of each tooth, where $y=\varepsilon$. The flux balance condition (1.9) is the graph generalization of the requirement that $a^\varepsilon\partial_x u^\varepsilon$ be continuous for it to have a weak derivative as a function on $\mathbb{R}$. We refer to §2.1 below for a discussion of diffusions on graphs.

It is convenient to change variables $y\mapsto y/\varepsilon$ so that the rescaled comb becomes

$$\Omega_\varepsilon \stackrel{\text{def}}{=} \big(\mathbb{R}\times\{0\}\big)\cup\big(\varepsilon\mathbb{Z}\times[0,1)\big)\,.$$

With this rescaling, our one-dimensional model becomes

$$\partial_t u^\varepsilon - \frac{1}{2}\partial_x^2 u^\varepsilon = 0 \qquad \text{when } x\notin\varepsilon\mathbb{Z},\ y=0\,, \tag{1.10}$$

$$\partial_t u^\varepsilon - \frac{1}{2}\partial_y^2 u^\varepsilon = 0 \qquad \text{when } x\in\varepsilon\mathbb{Z},\ y\in(0,1)\,, \tag{1.11}$$

with boundary conditions

$$\frac{\varepsilon}{2}\partial_y u^\varepsilon + \frac{1}{2}\partial_x^+ u^\varepsilon - \frac{1}{2}\partial_x^- u^\varepsilon = 0 \qquad (x,y,t)\in\varepsilon\mathbb{Z}\times\{0\}\times(0,T)\,, \tag{1.12}$$

$$\partial_y u(x,y,t) = 0 \qquad (x,y,t)\in\varepsilon\mathbb{Z}\times\{1\}\times(0,T)\,. \tag{1.13}$$

For initial data, we take

$$u^\varepsilon(x,y,0) = U_0(x,y), \quad (x,y)\in\Omega_\varepsilon,$$

where $U_0\in C^0(\mathbb{R}\times[0,1])$.

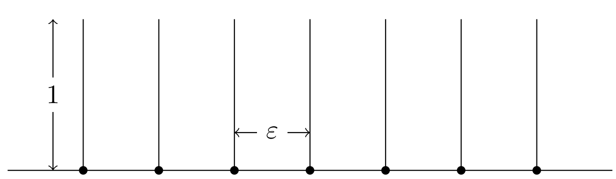

FIGURE 2. Image of the rescaled comb Ω_ε. The model satisfies the heat equation away from the base of the teeth and the flux balance condition (1.12) on $\varepsilon\mathbb{Z}\times\{0\}$.

The stochastic process associated to (1.10)-(1.13) is a diffusion on Ω_ε for which (1.10)-(1.13) is the Kolmogorov equation. Our goal is to describe this process and its limit behavior from a probabilistic point of view and to make a connection between (1.10)-(1.13) and some recent work on trapped random walks [BAC+15]. We will denote the diffusion process by $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$. Away from the vertices $\varepsilon\mathbb{Z} \times \{0, 1\}$ the process is simply a Brownian motion along the teeth or along the spine, according to (1.10) and (1.11). Corresponding to the Neumann condition (1.13), the process is reflected at the free end of each tooth, at the vertices $\varepsilon\mathbb{Z} \times \{1\}$. At the junction points $\varepsilon\mathbb{Z} \times \{0\}$, however, the flux balance condition (1.12) implies (roughly speaking) that the process enters the teeth with probability $\varepsilon/(2+\varepsilon)$ in the sense of Corollary 2.3, and continues in the spine otherwise (see Corollary 2.3). So, as $\varepsilon \to 0$, excursions into the teeth become less likely. On the other hand, the density of the junction points increases as $\varepsilon \to 0$, so that while the process is on the spine it very frequently meets a junction point. The balance of these two dynamics leads to interesting limit behavior because the process spends a non-vanishing amount of time in the teeth as $\varepsilon \to 0$.

To describe the diffusion process and its limit behavior more precisely, we consider two complimentary points of view. First, we define $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ as a solution to a certain system of SDEs, involving a constraint on the local time of Z^ε at the junction points. With $\varepsilon > 0$, the process behaves like a skew Brownian motion (see Remark 2.4 or [Lej06]), skewed at the junction points $\varepsilon\mathbb{Z} \times \{0\}$. The second approach to describing $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ involves Itô's excursion theory for Brownian motion [Itô72, PY07]. From this point of view, the process consists of infinitely many excursions from the junction points: excursions into the teeth, and excursions across the spine. We can view the vertical excursions into the teeth as "traps" for the macroscopic horizontal diffusion. In this way we can study the homogenization in a framework for trapped 1-D random walks developed in [BAC+15].

Having defined the process in these two ways, we then consider the limit behavior of Z^ε as $\varepsilon \to 0$, from the SDE point of view and from the excursion point of view. We show that as $\varepsilon \to 0$ the tooth component Y^ε converges to a Brownian motion Y on $[0, 1]$ which is sticky at $y = 0$. The spine component $X^\varepsilon(t)$ converges to a time-changed Brownian motion on $\mathbb{R}$, and the time change is described in terms of the local time of Y at $y = 0$. Specifically, our main result is the following:

Theorem 1.2. *As $\varepsilon \to 0$, $(X^\varepsilon, Y^\varepsilon)$ converges in law to (X, Y), where Y_t is a doubly reflected Brownian motion on $[0, 1]$ which is sticky at 0 with parameter $\mu = 1/2$ and*

$$X_t = \bar{W}_{2L^Y_t(0)}\,,$$

where $\bar{W}$ be a Brownian motion on $\mathbb{R}$ that is independent of $\bar{B}$ and $L^Y_t(0)$ is the local time of Y at 0.

A discussion of sticky Brownian motion can be found below in §2.2 but a quick construction can be stated as follows. Let $\bar{B}_t$ be a doubly reflected Brownian motion on $[0, 1]$ and define

$$\varphi(s) \stackrel{\text{def}}{=} s + 2L^{\bar{B}}_s(0)\,.$$

Let T be the inverse of φ:

$$T(t) = T_t \stackrel{\text{def}}{=} \varphi^{-1}(t) = \inf\{s \geqslant 0 \mid \varphi(s) \geqslant t\}\,.$$

In the definition of X_t, the process $L_t^Y(0)$ is defined by

$$L_t^Y(0) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t \mathbf{1}_{\{0 < Y_s \leqslant \delta\}} \, ds$$

where the strict inequality $0 < Y_s$ in the integrand is crucial. The times at which the sticky process Y is equal to 0 is either empty or has positive measure, since

$$\int_0^t \mathbf{1}_{\{Y_s = 0\}} \, ds = 2L_{T_t}^{\bar{B}}(0) = 2L_t^Y(0) \,.$$

(See §2.2 for a discussion of sticky Brownian motion.)

The second toy model is a two-dimensional diffusion which is a "fattened up" version of the previous comb model. For the fat comb model, the domain $\Omega_\varepsilon \subset \mathbb{R}^2$ is the connected and unbounded open set defined by

$$\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 \;\; | -\varepsilon < y < \mathbf{1}_{B(\varepsilon\mathbb{Z}, \varepsilon^2/2)}(x)\} \,. \tag{1.14}$$

Here $\varepsilon \in (0, 1]$ and ε^2 are the width of the spine and teeth respectively, $B(\varepsilon\mathbb{Z}, \varepsilon^2/2)$ denotes the $\varepsilon^2/2$ neighborhood of $\varepsilon\mathbb{Z}$ in $\mathbb{R}$, and $\mathbf{1}$ is the indicator function. The teeth have height one (for simplicity), and they are joined to the spine at spacing ε. See Figure 3 below. Let $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ be a standard Brownian motion in Ω_ε that is reflected internally at the boundary $\partial\Omega_\varepsilon$. The process Z^ε may travel within the spine or wander into the teeth, but when ε is small, the relatively narrow width of the teeth hinders the passage of Z^ε from the spine into a tooth. On the other hand, the teeth occur at high density along the spine; because of this and because the spine is also narrow, Z^ε encounters the teeth quite often. The balance of these dynamics leads to an interesting limit as $\varepsilon \to 0$, where the process spends non-vanishing amount of time in both the spine and the teeth. In fact, we will show that Theorem 1.2 also holds for the process $(X^\varepsilon, Y^\varepsilon)$ in the fat comb model with the same limit (X, Y) as the thin comb.

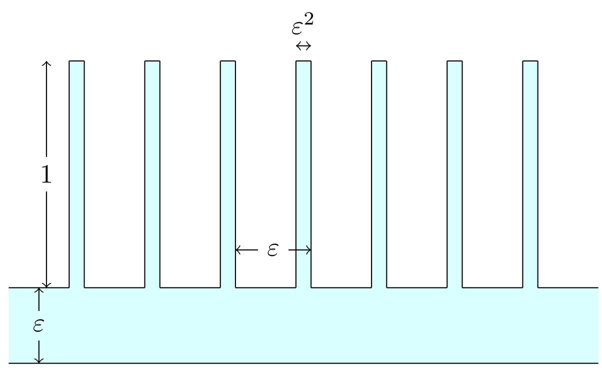

FIGURE 3. Image of the fat comb Ω_ε. The teeth have width ε^2 and height 1, the spine has width ε and ε spacing between teeth.

2. Preliminaries

2.1. **Diffusion on Graphs.** Let Γ be a connected graph consisting of vertices $\{O_k\}_{k=1}^M$ and edges $\{I_j\}_{j=1}^N$ embedded in Euclidean space. Suppose one would like

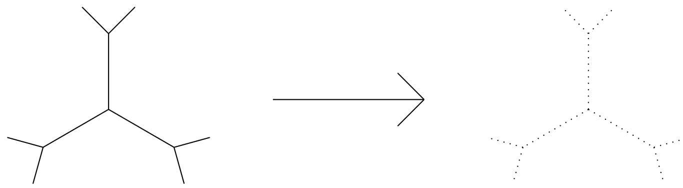

FIGURE 4. The left is a graph Γ and on the right is the corresponding discretization. The spacing between adjacent nodes is ε.

to consider a Brownian motion B on Γ, meaning $B_t \in \Gamma$ for all $t \geqslant 0$ and B behaves like a Brownian motion on each edge of the graph. If we start B in the interior of one of the edges I_j, then treating the edge like a closed interval in $\mathbb{R}$, we have no problem talking about the dynamics of B_t up until the stopping time $\tau = \inf\{t : B_t \in \partial I_j\}$ when B_t hits one of the vertices O_k. To continue, we need to decide on the behavior of the process at the vertex O_k.

The issue can be clarified by considering a discrete random walk model which approximates this diffusion on each edge. We divide each edge I_j of the graph into $\frac{1}{\varepsilon}|I_j|$ nodes with spacing ε and define a random walk S_k^ε which is is symmetric between the two neighbors for each node in the interior of an the edge. Then if we define

$$S_t^\varepsilon = S_k^\varepsilon \quad \text{for } t \in [\varepsilon^2 k, \varepsilon^2(k+1)),$$

then if S_t^ε converges as $\varepsilon \to 0$, the limiting process B will behave like a Brownian motion on each edge by the standard discrete approximation of Brownian motion. At the vertices O_k, there are as many neighbors as incident edges and it turns out that if we assign unequal jump probabilities, this will show up in the limiting process. These jump probabilities are directly reflected in the domain of the generator of the limiting process B_t. In fact, the generator condition was the original approach taken to identify the different possible diffusions on Γ.

On each edge I_j we consider an uniformly elliptic operator parametrized by the arc length of the form

$$\mathcal{L}_i f = \frac{1}{2}\sigma_i^2(z)\partial_z^2 f(z) + b_i(z)\partial_z f(z),\ z > 0$$

where $\sigma_i(z) \geqslant \alpha > 0$ for all $i = 1, \ldots, N$. The existence of a diffusion on Γ with the above generator on each edge was developed by Friedlin and Wentzell in [FW93] using the Hille-Yosida theorem. By considering these operators on each edge together as one operator on the graph, we define $Af(z) = \mathcal{L}_j f(z)$ on I_j with $A : \mathcal{D}(A) \to C^0(\Gamma)$ where $\mathcal{D}(A)$ is chosen appropriately. Note that this continuity means we require $\mathcal{L}_j f(O_k) = \mathcal{L}_i f(O_k)$ for $f \in \mathcal{D}(A)$ and $I_i, I_j \sim O_k$. To establish uniqueness, of the differential equation on Γ, an appropriate gluing condition is needed at each vertex O_k of the form

$$\tilde{\alpha}_k A f(O_k) = \sum_{I_j \sim O_k} \alpha_{kj} D_j f(O_j)\,. \tag{2.1}$$

where $D_j f(O_k)$ is the outward derivative on I_j at O_k and $\tilde{\alpha}_k, \alpha_{kj} \geqslant 0$ are not all 0. We can interpret $\tilde{\alpha}_k$ as a "stickiness" parameter which causes the process Z to spend non-trivial time at O_k and we discuss in §2.2 below for the single edge case of a reflected Brownian motion. This is summarized in the following theorem.

Theorem 2.1 (Theorem 3.1 in [FS00]). *The operator A is the infinitesimal generator of a Feller continuous strong Markov process $Z(t)$ on Γ with almost sure continuous sample paths. In addition, $\tilde{\alpha}_k = 0$ if and only if $\{t : Z(t) = O\}$ has Lebesgue measure 0.*

Following this, Freidlin and Sheu developed an Itô formula on graphs in the case $\tilde{\alpha}_k = 0$ for all $k = 1, \dots, M$ which can be used to establish a stochastic calculus in this setting.

Lemma 2.2 (Lemma 2.3 in [FS00]). *Consider a graph diffusion $Z(t)$ which in a neighborhood Γ_O of a vertex O has generator A defined by*

$$Af = \mathcal{L}_i f = \frac{1}{2}\sigma_i^2(z)\partial_z^2 f(z) + b_i(z)\partial_z f(z),\ z > 0$$

on edge I_i and has domain $\mathcal{D}(A) = \{f \in C_b^\infty(\Gamma_O) \mid \rho(f) = 0\}$ where

$$\rho(f) = \sum_{i=1}^{N} \alpha_i D_i f(O)$$

where $C_b^\infty(\Gamma_O)$ is the set of functions which are C_b^∞ on each of the branches and continuous on Γ_O. Write $Z(t) = (z(t), i(t))$ with $z(t) = d(Z(t), O)$ where d is the graph distance and $i(t)$ is a label for the branch on which Z resides at time t. Then for $\tau = \inf\{s \mid Z(s) \notin \Gamma_O\}$,

$$F(Z(t \wedge \tau)) = F(Z(0)) + \int_0^{t\wedge\tau} \sigma_{i(s)}(z(s))\frac{dF_{i(s)}}{dz}(z(s))dW_s + \int_0^{t\wedge\tau} AF(Z(s))ds + \rho(F)\ell(t\wedge\tau)$$

where ℓ is a non-decreasing and only increases when Z at O.

As a corollary we obtain a nice interpretation of the gluing condition. First suppose we normalize (2.1) so that

$$\sum_{I_j \sim O_k} \alpha_{kj} = 1\,,$$

then we can think of α_{kj} as the probability of entering I_j after hitting O_k in the following sense,

Corollary 2.3 (Corollary 2.4 in [FS00]). *Let Γ_O be a neighborhood of a vertex O with incident edges $\{I_j\}_{j=1}^N$ and suppose the gluing condition of A at O has the form*

$$0 = \sum_{i=1}^{N} \alpha_i D_i f(O)\,.$$

Let θ^δ, $\delta > 0$ be the stopping time defined by

$$\theta^\delta = \inf\{t : z(t) \geqslant \delta\}\,.$$

Then

$$\lim_{\delta\to 0} P^O(i(\theta^\delta) = j) = \alpha_j$$

Remark 2.4. This phenomenon can also be observed in a diffusion on $\mathbb{R}$ by viewing $\mathbb{R}$ as a graph with two edges and a vertex at 0. A diffusion X_t on $\mathbb{R}$ that behaves like a Brownian motion when $X_t \neq 0$ but starting from $x = 0$ has probability p of taking an excursion to the right and $1-p$ to the left had been known long before the graph diffusion case was understood and is called a *skew Brownian motion.* For a thorough discussion of the history and a number of constructions of skew Brownian motion, see the survey by Lejay [Lej06].

What can we say about the process ℓ_t in Lemma 2.2? First, by applying Lemma 2.2 to $F_j(Z) = d(Z, O)\mathbf{1}_{I_j}(Z)$ where d is the graph distance,

$$F_j(Z(t)) = F_j(Z(0)) + \int_0^t \sigma_j(z(s))\mathbf{1}_{\{i(s)=j\}}\, dW_s + \int_0^t b_j(z(s))\mathbf{1}_{\{i(s)=j\}} ds + \alpha_j \ell(t)\,. \tag{2.2}$$

By a mollification argument we can extend the Itô formula to functions which are $C_b^1(\Gamma_O)$ with $\partial_Z^2 F_j(z) \in L^\infty$. So applying this extended version of Lemma 2.2 to

$$G_j^\delta(Z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{1}{\delta} z^2 & \text{if } 0 \leqslant z < \delta,\ Z \in I_j\ , \\ z - \dfrac{\delta}{2} & \text{if } z > \delta,\ Z \in I_j\ , \\ 0 & \text{otherwise} \end{cases}$$

gives the equation

$$G_j^\delta(Z(t)) = G_j^\delta(Z(0)) + \int_0^t \sigma_j(z(s))\partial_z G_j^\delta(Z(s))\mathbf{1}_{\{i(s)=j\}}\, dW_s + \frac{1}{2\delta}\int_0^t \sigma_j^2(z(s))\mathbf{1}_{\{0<F_j\left(Z(s)\right)<\delta\}}\, ds + \int_0^t b_j(z(s))\partial_z G_j(Z(s))\mathbf{1}_{\{i(s)=j\}} ds.$$

One can show that all the above terms converge almost surely to their corresponding terms in (2.2) as $\delta \to 0$ and hence

$$\alpha_j \ell_t = \lim_{\delta\to 0} \frac{1}{2\delta}\int_0^t \sigma_j^2(z(s))\mathbf{1}_{\{0<F_j\left(Z(s)\right)<\delta\}}\, ds = L_t^{Z_j}(O)\,, \tag{2.3}$$

where $L^{Z_j}(O)$ is the local time at O of the process on I_j. Hence

$$\ell_t = \sum_{j=1}^N L^{Z_j}(O) = \lim_{\delta\to 0}\int_0^t \sigma_{i(s)}(z(s))\mathbf{1}_{\{d(z(s),O)<\delta\}}\, ds$$

is the local time at O of the joint process on all branches. One interesting consequence of this formula is that the local time at O at two adjacent edges I_j and I_i stays in direct proportion almost surely, i.e. $\frac{1}{\alpha_j} L_t^{Z_j}(O) = \frac{1}{\alpha_i} L_t^{Z_i}(O)$ for all $t \geqslant 0$. We can understand this "local time balance" using Corollary 2.3. When $Z(t)$ hits the vertex O, it makes infinitely many small excursions into each edge. The corollary says that for any excursion of length at least $\delta > 0$, the probability that it's into branch I_j tends to α_j as $\delta \to 0$. The time spent by the process $Z(t)$ near O is dominated by the small excursions and the number of excursions of length at least δ tends to infinity. Thus by the Law of Large numbers, the proportion of time spent on each edge within a distance δ of O tends to the average probability of being on each edge I_j which is α_j.

2.2. **Sticky Brownian Motion.** Consider a random walk S_k on $\varepsilon\mathbb{Z}$ which jumps left and right with probability $1/2$ when $S_k \neq 0$. When $S_k = 0$, the walk stays in place with probability $1 - \mu\varepsilon$ and jumps left and right with probability $\mu\varepsilon/2$ each for some $\mu > 0$. If we define

$$S_t^\varepsilon = S_k^\varepsilon \quad \text{for } t \in [\varepsilon^2 k, \varepsilon^2(k+1)) \,,$$

then S_t converges to a process Y_t. By the standard discrete approximation of Brownian motion, Y_t behaves like a Brownian motion when $Y_t \neq 0$, but the scaling for the escape probability leads to a nontrivial amount of time being spent at 0. This limiting process is known as *Sticky Brownian motion*. As we will see, the process is characterized by the relation

$$\int_0^t \mathbf{1}_{\{Y_s=0\}} \, ds = \frac{1}{\mu} L_t^Y(0)$$

and so the time spent at 0 of a sticky Brownian motion has positive Lebesgue measure. It was first discovered by Feller in a paper looking for all Strong Markov processes X_t on $[0, \infty)$ which behave like Brownian motion on $(0, \infty)$ [Fel52]. Using the Hille - Yosida theorem, he was able to prove the existence of a process with generator $A = \frac{1}{2}\partial_y^2$ with domain $f \in \mathcal{D}(A)$

$$f'(0) = \frac{1}{2\mu} f''(0) \,. \tag{2.4}$$

This is the one edge variant of equation (2.1) with $\tilde{\alpha}_k \neq 0$.

A direct construction of Sticky Brownian motion was described by Itô and McKean [IM63]. We begin with a standard 1-D Brownian motion B_t and define $\varphi(s) \stackrel{\text{def}}{=} s + (1/\mu) L_s^B(0)$. Let T be the inverse of φ:

$$T(t) = T_t \stackrel{\text{def}}{=} \varphi^{-1}(t) = \inf\{s \geqslant 0 \mid \varphi(s) \geqslant t\} \,.$$

Then $Y_t = B_{T_t}$ is a sticky Brownian motion. Note that the zero set of Y_t is still nowhere dense since the time change T_t is a homeomorphism. This implies a nonsingular behavior of $L_t^Y(0)$ in the sense that

$$\lim_{t \to 0} \frac{1}{t} \boldsymbol{E}^0 \left[L_t^Y(0) \right] = \mu < \infty$$

which is the key to calculating the condition (2.4) for $\mathcal{D}(A)$.

2.3. **Fractional Time Equations and Time Changed Brownian Motion.** Phenomenon which are diffusive in nature but do not spread at the usual rate as Brownian motion have been been observed in a plethora of physical systems (see the survey [MK00] and the references therein for an extensive historical treatment). Such processes are known as *anomalous diffusion* and are characterized as a diffusive process $X(t)$ whose variance has a non-linear relation with time, in contrast to that of Brownian motion whose variance grows linearly. The case where the variance grows sub-linearly, i.e $\boldsymbol{E}[X(t)^2]/t \to 0$ as $t \to \infty$ is known as *sub-diffusion.* Fractional time equations were used since the 1980's as a model for sub-diffusion [Bal85,Wys86,SW89] and the connection between these equations and what is now called the fractional-kinetics process was established in '02 by [MBSB02]. To describe the fractional-kinetics process, we introduce the concept of a subordinator.

A *subordinator* $S(t)$ is an increasing Lévy process, meaning a stochastic process stationary and independent increments with càdlàg paths almost surely. By an *inverse subordinator* we mean the right continuous inverse of S, namely $E(t) = \inf\{r : S(r) > t\}$. Subordinators can be characterized by their Laplace transforms which have a special form given by the Lévy-Khintchine formula. For any subordinator $S(t)$, there exists a unique associated measure μ on $(0, \infty)$ satisfying

$$\int_0^\infty (x \wedge 1)\, \mu(dx) < \infty$$

which is called the Lévy measure. The Laplace transform then satisfies

$$\boldsymbol{E}\left(e^{-\lambda S(t)}\right) = e^{-tf(\lambda)}$$

where the function f (called the Lévy exponent) can be written in the form

$$f(\lambda) = \kappa\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right)\, \mu(dx)\,.$$

If the Lévy measure of $S(t)$ has infinite mass, i.e. $\mu(0, \infty) = \infty$ or if $\kappa > 0$, then $S(t)$ is strictly increasing and hence $E(t)$ has continuous trajectories. For $\alpha \in (0, 1)$ a subordinator $S_\alpha(t)$ is called α-stable if $(1/t^{\frac{\alpha}{2}})S(t) \simeq S(1)$ in distribution. The fractional-kinetics process is a d-dimensional Brownian motion time changed by the inverse of an α-stable subordinator i.e. $X_{d,\alpha}(t) = B(E_\alpha(t))$. In this case the variance grows as a sub-linear power law, $\boldsymbol{E}[X^2_{d,\alpha}(t)] = ct^\alpha$. The fractional-kinetics process has also been found to as the limit of scaled random walks which have waiting times with heavy tails which makes it the continuous analogue for a number of discrete trapped random walk models [BAČ07, MS04].

Let $f \in C^1([0, \infty); \mathbb{R})$ and $\alpha \in (0, 1)$. The *Caputo fractional derivative of order* α is defined by

$$\partial_t^\alpha f(t) \stackrel{\text{def}}{=} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} (f(s) - f(0))\, ds = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds\,,$$

where Γ is the usual Gamma function. It was established in [MBSB02] that diffusions time changed by the inverse of an α-stable subordinator had transition density p which solves a fractional time diffusion equation of the form

$$\partial_t^\alpha p(x, t) = \mathcal{L} p(x, t)\,.$$

where $\mathcal{L}$ is the generator of the diffusion. This in turn gave stochastic representation formulas for such fractional diffusion equations. This in fact extends to any subordinator giving us a connection between "trapped Brownian motions" and a general class of fractional time equations. The variant of these results that we will use is as follows:

Theorem 2.5. *Suppose B_t is a d-dimensional Brownian motion and S_t is a subordinator and let*

$$f(\lambda) = \kappa\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right)\, \mu(dx)$$

be the Laplace exponent of S i.e. $\boldsymbol{E}[e^{-\lambda S(t)}] = e^{-tf(\lambda)}$. Suppose $\mu(0, \infty) = \infty$ or $\kappa > 0$ and let $w(x) = \mu(x, \infty)$. Denote the right continuous inverse of $S(t)$ by $E(t) = S^{-1}(t)$. Then $v(x, t) = \boldsymbol{E}^x\left[g(B(E(t)))\right]$ is a strong solution to

$$\kappa \partial_t v(x, t) + \partial_t^w v(x, t) = \frac{1}{2} \Delta v(x, t) \quad \text{for } t > 0, x \in \mathbb{R}^n \tag{2.5}$$

$$v(x,0) = g(x) \qquad\qquad \textit{for } x \in \mathbb{R}^n\,,$$

where ∂_t^w *is a generalization of the Caputo derivative defined by*

$$\partial_t^w v(x,t) \stackrel{\text{def}}{=} \int_0^t w(t-s)\partial_t v(x,s)\,ds\,.$$

Proof. We use the approach in [MS13] (also see [Che17]). Note that the assumption that $\mu(0,\infty) = \infty$ implies that $\boldsymbol{P}(S(r) = t) = 0$ for all $r, t \geqslant 0$. The inverse subordinator $E(t)$ has a density on $[0,\infty)$ which we denote by $e(r,t)$. Let $u(x,t) \stackrel{\text{def}}{=} \boldsymbol{E}^x\,[g(B(t))]$ which satisfies

$$\partial_t u(x,t) = \frac{1}{2}\Delta u(x,t)\,.$$

We denote the Laplace transforms in time for v,e and u by $\hat{v}(x,\lambda)$, $\hat{e}(x,\lambda)$ and $\hat{u}(x,\lambda)$ respectively. We first compute $\hat{e}(r,\lambda)$:

$$\begin{aligned}
\hat{e}(r,\lambda) &= \int_0^\infty e^{-\lambda t} e(r,t)\,dt \\
&= \int_0^\infty e^{-\lambda t}\frac{d}{dr}\boldsymbol{P}(E(t) \leqslant r)\,dt \\
&= \frac{d}{dr}\int_0^\infty e^{-\lambda t}\boldsymbol{P}(t \leqslant S(r))\,dt \\
&= \frac{1}{\lambda}\frac{d}{dr}\left[1 - \boldsymbol{E}\left(e^{-\lambda S(r)}\right)\right] \\
&= -\frac{d}{dr}\frac{1}{\lambda}e^{-rf(\lambda)}\,.
\end{aligned}$$

Taking Laplace transforms of the diffusion equation we see $\hat{u}$ satisfies

$$\lambda\hat{u}(x,\lambda) - g(x) = \frac{1}{2}\Delta\hat{u}(x,\lambda)\,. \tag{2.6}$$

Then for $\hat{v}$ we can write

$$\begin{aligned}
\hat{v}(x,\lambda) &= \int_0^\infty e^{-\lambda t}\int_0^\infty u(x,r)e(r,t)\,dr\,dt \\
&= \int_0^\infty u(x,r)\hat{e}(r,\lambda)\,dr \\
&= \int_0^\infty u(x,r)\frac{f(\lambda)}{\lambda}e^{-rf(\lambda)}\,dr \\
&= \frac{f(\lambda)}{\lambda}\hat{u}(x,f(\lambda))\,.
\end{aligned}$$

Plugging $f(\lambda)$ into (2.6) and using the above yields

$$\lambda\hat{v}(x,\lambda) - g(x) = \frac{\lambda}{f(\lambda)}\frac{1}{2}\Delta\hat{v}(x,\lambda)$$

which we can rearrange to obtain

$$\kappa\lambda\hat{v}(x,\lambda) - \kappa g(x) + (f(\lambda) - \kappa\lambda)\hat{v}(x,\lambda) - \left(\frac{f(\lambda) - \kappa\lambda}{\lambda}\right)g(x) = \frac{1}{2}\Delta\hat{v}(x,\lambda)\,.$$

Using the relation $\lambda\hat{w}(\lambda) = f(\lambda) - \kappa\lambda$ which can be directly shown using integration by parts, it suffices to show

$$(\partial_t^w v)^\wedge(x,\lambda) = \hat{w}(\lambda)(\lambda\hat{v}(x,\lambda) - g(x))\,.$$

We verify this directly with the following computation:

$$
\begin{aligned}
(\partial_t^w v)^\wedge(x,\lambda) &= \int_0^\infty e^{-t\lambda} \int_0^t w(t-s)\partial_t v(x,s)\,ds\,dt \\
&= \int_0^\infty e^{-s\lambda}\partial_t v(x,s) \int_s^\infty e^{-(t-s)\lambda} w(t-s)\,ds\,dt \\
&= \hat{w}(\lambda)(\lambda\hat{v}(x,\lambda) - g(x))\,.
\end{aligned}
$$

□

Given the solvability of the homogeneous equation, the inhomogeneous equation can be solved using an analog of Duhamel's principle [Uma12, US06].

Lemma 2.6 (Duhamel's Principle for Fractional Time Equations)**.** *For $\kappa > 0, s \geqslant 0$, let $\tilde{v}_s$ be a solution to the equation*

$$
\kappa\partial_t\tilde{v}_s(x,t) + \frac{1}{2}\partial_t^w\tilde{v}_s(x,t) - \frac{1}{2}\Delta\tilde{v}_s(x,t) = 0\,, \qquad \text{for } t > s\,, \tag{2.7}
$$

$$
\tilde{v}_s(x,s) = \Big(\kappa I + \frac{1}{2}\mathcal{I}_s^w\Big)^{-1} g(x,\cdot)\,. \tag{2.8}
$$

Here $\mathcal{I}_\cdot^w$ is the integral operator with kernel w defined by

$$
\mathcal{I}_t^w h = \int_0^t w(t-s)h(s)\,ds\,.
$$

Then the function v defined by

$$
v(x,t) \stackrel{\text{def}}{=} \int_0^t \tilde{v}_s(x,t)\,ds\,,
$$

is a strong solution to the inhomogeneous equation

$$
\begin{aligned}
\kappa\partial_t v(x,t) + \partial_t^w v(x,t) - \frac{1}{2}\Delta v(x,t) &= g(x,t) \qquad \text{for } t > 0, x \in \mathbb{R}^n \\
v(0) &= f(x) \qquad x \in \mathbb{R}^n\,.
\end{aligned} \tag{2.9}
$$

.

Proof of lemma. Since $\mathcal{I}^w$ is a positive compact operator, the operator $(\kappa I + \mathcal{I}^w/2)$ is invertible, ensuring the initial condition (2.8) can be satisfied. For convenience, define $\tilde{v}_s(x,r) = \tilde{v}_s(x,s)$ when $r < s$. Now, it is a direct computation to verify that the function $v(x,t)$ solves equation (2.9),

$$
\begin{aligned}
\kappa\partial_t v(x,t) &= \kappa\partial_t \int_0^t \tilde{v}_s(x,t)\,ds \\
&= \kappa\tilde{v}_t(x,t) + \kappa\int_0^t \partial_t\tilde{v}_s(x,t)\,ds \\
&= \kappa\tilde{v}_t(x,t) + \frac{1}{2}\Delta v(x,t) - \frac{1}{2}\int_0^t \partial_t^w\tilde{v}_s(x,t)\,ds\,.
\end{aligned}
$$

We have,

$$
\begin{aligned}
\partial_t^w v(x,t) &= \int_0^t w(t-r)\partial_r\Big(\int_0^r \tilde{v}_s(x,r)\,ds\Big)\,dr \\
&= \int_0^t w(t-r)\Big(\tilde{v}_r(x,r) + \int_0^r \partial_r\tilde{v}_s(x,r)\,ds\Big)\,dr
\end{aligned}
$$

$$= I_t^w \tilde{v}.(x,\cdot) + \int_0^t \int_s^t w(t-r)\partial_r \tilde{v}_s(x,r)\,dr\,ds$$
$$= I_t^w \tilde{v}.(x,\cdot) + \int_0^t \partial_t^w \tilde{v}_s(x,t)\,ds\,,$$

where the last equality uses $\partial_r \tilde{v}_s(x,r) = 0$ for $r < s$. If we substitute this into (2.9), we see

$$\kappa\partial_t v + \frac{1}{2}\partial_t^w v = \kappa v_t(x,t) + \frac{1}{2}I_t^w \tilde{v}.(x,\cdot) + \frac{1}{2}\Delta v = g + \frac{1}{2}\Delta v$$

and hence $v(x,t)$ solves the inhomogeneous problem. □

2.4. **Trapped Brownian Motion.** In [BAC+15], the authors introduced a general framework to describe trapped Brownian motions on $\mathbb{R}$ and characterized all possible scaling limits of a fixed (random) trap structure. A *Lévy trap measure* on $\mathbb{R}\times[0,\infty)$ is a random measure such that $\mu(A\times[0,t])$ is a Lévy process for all bounded $A\subset\mathbb{R}$. We say that μ has independent increments if $\mu(E)$ is independent from $\mu(F)$ whenever $E\cap F=\emptyset$. Let $B(t)$ be a standard 1D Brownian motion and let ψ be a time-change of the form

$$\psi(t) = \inf\big\{s>0 \,\big|\, \phi[\mu,B]_s > t\big\}\,,$$

where

$$\phi[\mu,B]_s = \mu\left(\{(x,\ell)\in\mathbb{R}\times[0,\infty) \mid L_s^B(x)\geqslant \ell\}\right)\,,$$

$L^B(x)$ is the local time of B at x and μ is a Lévy trap measure with independent increments on $\mathbb{R}\times[0,\infty)$. Then $B(\psi(t))$ is known as a trapped Brownian motion, denoted by $B[\mu]$.

When μ is Lebesgue measure on $\mathbb{R}\times[0,\infty)$, then $\phi[\mu,B]=t$, $\psi(t)=t$ and hence the trapped Brownian motion is just a Brownian motion. If the measure μ has an atom at (x,ℓ) of mass $r>0$, then $B(\psi(t))$ is trapped at x for a time r at the moment its local time at x exceeds ℓ.

The main result in [BAC+15] is the characterization of all scaling limits. Consider such a trapped Brownian motion $X_t = B(\psi(t))$ and then scale it to obtain the process

$$X_t^\varepsilon = \varepsilon X_{\rho(\varepsilon)^{-1}t}$$

where $\rho(\varepsilon)$ is an arbitrary function. Then if X_t^ε converges in law to some process U_t, then U is either a Brownian motion or in another class of trapped Brownian motion which the authors call FK-SSBM mixtures which we now describe. Let $\mathcal{F}^\star$ be the set of all Laplace exponents of subordinators equipped with the topology of pointwise convergence. Then let $\mathbb{F}$ be a σ finite measure on $\mathcal{F}^\star$ and (x_i,f_i) be a Poisson point process on $\mathbb{R}\times\mathcal{F}^\star$ with intensity measure $dx\times\mathbb{F}$. For each i, let S^i be an independent subordinator (from each other and from B) with Laplace exponent f_i. Define

$$\phi_t = \sum_i S^i(L_t^B(x_i)) + V^\gamma(t)$$

where V^γ is a γ-stable subordinator for some $\gamma\in(0,1)$. Then an FK-SSBM mixture is a process $B(\psi(t))$ where

$$\psi(t) = \inf\big\{s>0 \,\big|\, \phi_s > t\big\}\,,$$

where ϕ is a subordinator as above. The case where $V_t^\gamma = 0$ is called a spatially subordinated Brownian motion and the case where $\mathbb{F}=0$ is the fractional-kinetics process from §2.3 and hence the name FK-SSBM mixture.

The thin comb model which we introduced in the previous section can be viewed in the framework of trapped Brownian motions. In §5 we provide a proof of the main theorem using this theory. As we will see, the type of scaling in the comb model is of a different character than can be found in the above theorem and the limiting trap structure is not one of their scaling limits. What we will use from [BAC+15] is the following theorem which says that convergence of the trap measures is enough to prove convergence of the trapped Brownian motions.

Theorem 2.7. *Let μ^ε, $\varepsilon > 0$ be a family of Lévy trap measures with a.s. infinite mass and let B be a Brownian motion independent from them. Assume that as $\varepsilon \to 0$, μ^ε converges vaguely in distribution to a dispersed, dense, a.s. infinite random measure μ. Then the corresponding trapped Brownian motions $B[\mu^\varepsilon]$ converge to $B[\mu]$ in distribution on $D([0,\infty))$.*

2.5. **Reflected Diffusions.** Let $Z_t^\varepsilon = (X_t^\varepsilon, Y_t^\varepsilon)$ be the "fat comb diffusion" i.e. the reflected Brownian motion on the fat comb domain Ω^ε (see (1.14)). Existence and uniqueness for reflected Brownian motion was shown by Stroock and Varadhan [SV71] by finding solutions to certain "sub-martingale problems."

Differential equations for these reflected diffusions was developed first in convex domains and later in general Lipschitz domains using a variant of the Skorohod problem [Tan79, LS84].

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^d$ be a domain with uniformly Lipschitz boundary, with σ, b Lipschitz on Ω and $n : \partial\Omega \to \mathbb{R}^d$ the outward unit normal. Let W_t be a d-dimensional Brownian motion and $\mathcal{F}_t$ the augmented filtration generated by W. Then there exists a unique continuous $\mathcal{F}_t$ semi-martingale Z_t and increasing process ℓ_t such that $Z_t \in \bar{\Omega}$ for all $t \geqslant 0$:*

$$\ell_t = \int_0^t \mathbf{1}_{\partial\Omega}(Z_s)\, d\ell_s$$

and Z_t satisfies the SDE

$$dZ_t = \sigma(Z_s)dW_s + b(Z_s)ds - n(Z_s)d\ell_t\,.$$

In fact the process ℓ_t is the local time of Z on $\partial\Omega$ which can be defined by

$$L^Z(\partial\Omega) = \lim_{\delta\to 0} \frac{1}{2\delta} \int_0^t \mathbf{1}_{\{d(Z_s,\partial\Omega)<\delta\}}\, ds\,.$$

One needs a version of Itô's lemma in this setting, which allows for jumps in the gradient along hypersurfaces. The following theorem is likely in the literature, but we could not find good reference

Theorem 2.9. *Let Z_t be a continuous semi-martingale on $\mathbb{R}^d$ and let $f \in C^0(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \backslash M)$ where M be a $d-1$ dimensional orientable smooth manifold embedded in $\mathbb{R}^d$. Choose an orientation of M (equivalent to choosing a normal vector field $n(x)$ on M) and suppose ∇f has limits on either side of M denoted by $\nabla^- f$ and $\nabla^+ f$. Then the following SDE holds for $f(Z_t)$:*

$$df(Z_t) = \nabla^- f(Z_t)\cdot dZ_t + \frac{1}{2}\partial_i\partial_j f(Z_t)\mathbf{1}_{\{Z_t\notin M\}}\langle Z_i, Z_j\rangle + (\nabla^+ f(Z_t) - \nabla^- f(Z_t))\, dL_t^Z(M+) \tag{2.10}$$

where $\nabla^- f(x) = \nabla f(x)$ *for* $x \notin M$ *and* $L_t^Z(M+)$ *is the local time of* Z *on the positive side of* M*, i.e. let* $M^{\delta,+} = \{z \in \mathbb{R}^d : z = x + sn(x)$ *for some* $x \in M$ *and* $s \in (0,\delta)\}$*.*

$$L_t^Z(M+) = \lim_{\delta\to 0} \frac{1}{2\delta} \int_0^t \mathbf{1}_{\{Z_s \in M^{\delta,+}\}}\, ds\,.$$

This is a similar result as the generalized Itô's formula in 1D, which can be found in §3.6 of [KS91].

3. The Limit Process

3.1. **Properties of the Limit Process.** Before proving our main results in the following sections, we now give a more thorough description of the limit process $Z = (X, Y)$. One way to describe the limit process is via time-change of Brownian motions. Let $\bar{B}_t$ be a doubly reflected Brownian motion on $[0, 1]$. Let $L_s^{\bar{B}}(0)$ be the local time of $\bar{B}$ at 0, and define

$$\varphi(s) \stackrel{\text{def}}{=} s + 2L_s^{\bar{B}}(0), \quad s \geqslant 0\,.$$

Let T be the inverse of φ:

$$T(t) = T_t \stackrel{\text{def}}{=} \varphi^{-1}(t) = \inf\{s \geqslant 0 \mid \varphi(s) \geqslant t\}\,.$$

Let $\bar{W}$ be a Brownian motion on $\mathbb{R}$ that is independent of $\bar{B}$. Then the limit process is defined by

$$Y_t \stackrel{\text{def}}{=} \bar{B}_{T_t}\,, \qquad X_t \stackrel{\text{def}}{=} \bar{W}_{2L_t^Y(0)}\,. \tag{3.1}$$

and $(X_0, Y_0) \sim \mu$. Notice that Y_t is "sticky" at $\{y = 0\}$ in the sense of §2.2. Intuitively, we think of $\mathbb{R} \times \{0\}$ as the spine (in the limit), while $\mathbb{R} \times (0, 1]$ plays the role of a continuum of teeth. The process T_t may be interpreted as the time accumulated in the teeth, and $2L_t^Y(0)$ is the time accumulated in the spine. The limit process may also be described as a weak solution to the system

$$dX_t = \mathbf{1}_{\{Y_t=0\}}\, dB_t\,, \tag{3.2}$$

$$dY_t = \mathbf{1}_{\{Y_t\neq 0\}}\, dB_t - dL_t^Y(1) + dL_t^Y(0)\,, \tag{3.3}$$

$$\mathbf{1}_{\{Y_t=0\}}\, dt = 2\, dL_t^Y(0)\,, \tag{3.4}$$

with initial distribution μ, where B is a standard Brownian motion in $\mathbb{R}$. Existence and uniqueness for the system (3.2)–(3.4) can be established by arguments similar to those in [EP14].

Lemma 3.1. *For a given initial distribution* μ*, the system* (3.2)*–*(3.4) *has a unique weak solution, which is the process* $Z = (X, Y)$ *defined by* (3.1)*.*

Although this construction is well known (see [EP14] and references therein), for completeness we provide a proof of Lemma 3.1 below. One can alternately prove existence of a process Z satisfying (3.2)–(3.4) abstractly using the Hille-Yosida theorem, which we do in §3.2 for completeness.

Proof of Lemma 3.1. We adapt the approach in [EP14, Theorem 1]. By the Tanaka formula we have

$$\bar{B}_t = \tilde{B}_t + L_t^{\bar{B}}(0) - L_t^{\bar{B}}(1)\,, \tag{3.5}$$

where $\tilde{B}$ is a Brownian motion. Since T_t is a continuous and increasing time change, $\tilde{B}_{T_t}$ is still a continuous martingale, $L^Y_t(0) = L^{\bar{B}}_{T_t}(0)$ and $L^Y_t(1) = L^{\bar{B}}_{T_t}(1)$. Note first

$$\int_0^t \mathbf{1}_{\{Y_s=0\}}\, ds = \int_0^t \mathbf{1}_{\{\bar{B}_{T_s}=0\}}\, d\varphi(T_s) = \int_0^{T_t} \mathbf{1}_{\{\bar{B}_s=0\}}\, d\varphi(s).$$

Then since $\{t \mid \bar{B}_t = 0\}$ has Lebesgue measure 0 and $L^{\bar{B}}_t$ only increases on this set, we decompose $\varphi(s) = s + 2L^{\bar{B}}_s$ to obtain

$$\int_0^{T_t} \mathbf{1}_{\{\bar{B}_s=0\}}\, d\varphi(s) = 2\int_0^{T_t} \mathbf{1}_{\{\bar{B}_s=0\}} dL^{\bar{B}}_s(0) = 2L^{\bar{B}}_{T_t}(0) = 2L^Y_t(0)\,,$$

which implies (3.4). Notice that since $2L^Y_t(0)$ is independent of $\bar{W}$, X_t is a martingale with quadratic variation

$$\langle X\rangle_t = 2L^Y_t\,.$$

In addition we have

$$\langle \tilde{B}_T\rangle_t = T_t\,.$$

Thus, by Lévy's criterion, the process B defined by

$$B_t \stackrel{\text{def}}{=} \tilde{B}_{T_t} + \bar{W}_{2L^Y_t(0)} \tag{3.6}$$

is a Brownian motion. Now (3.2)–(3.3) follow from (3.4), (3.6) and the fact that

$$\int_0^t \mathbf{1}_{\{Y_s=0\}}\, d\tilde{B}_{T_s} = 0 \quad \text{and} \quad \int_0^t \mathbf{1}_{\{Y_s\neq 0\}}\, dX_s = 0\,.$$ □

For later use in our proofs of the main results, we now study the generator of the process Z. Let $\Omega_0 = \mathbb{R}\times[0,1)$, and define the operator A by

$$A \stackrel{\text{def}}{=} \frac{1}{2}\partial_y^2\,. \tag{3.7}$$

We define the domain of A, $\mathcal{D}(A)$, to be the set of all functions $g \in C_0(\Omega_0)\cap C^2_b(\Omega_0)$ such that

$$\partial_y g(x,1) = 0\,, \qquad \text{and} \qquad \partial_x^2 g(x,0) + \partial_y g(x,0) = \partial_y^2 g(x,0)\,. \tag{3.8}$$

We claim that this operator is exactly the generator of the process Z.

Lemma 3.2. *The generator of Z is the operator A with domain $\mathcal{D}(A)$.*

Proof of Lemma 3.2. To compute the generator of Z, we choose $g \in \mathcal{D}(A)$ and apply Itô's formula to obtain

$$\begin{aligned} g(X_t,Y_t) = g(X_0,Y_0) &+ \int_0^t \partial_x g(X_s,Y_s) dX_s + \int_0^t \partial_y g(X_s,Y_s)\, dY_s \\ &+ \int_0^t \partial_x^2 g(X_s,Y_s)\, dL^Y_s + \frac{1}{2}\int_0^t \partial_y^2 g(X_s,Y_s)\, dT_s\,. \end{aligned}$$

Taking expectations gives

$$\begin{aligned} \boldsymbol{E}^{(x,y)}\Big[g(X_t,Y_t) - g(x,y)\Big] &= \boldsymbol{E}^{(x,y)}\Big[\int_0^t \partial_y g(X_s,Y_s)\, dY_s\Big] \\ &+ \boldsymbol{E}^{(x,y)}\Big[\int_0^t \partial_x^2 g(X_s,Y_s)\, dL^Y_s + \frac{1}{2}\int_0^t \partial_y^2 g(X_s,Y_s)\, dT_s\Big]\,. \end{aligned} \tag{3.9}$$

Note that by by (3.4) we know $L^Y_t \leqslant t/2$.

Now for $y \in (0,1)$ we know Y is a Brownian motion before it first hits 0 or 1, and hence $\lim_{t\to 0} \boldsymbol{P}^y(L_t^Y \neq 0) = 0$. Moreover by definition of T, we know $T_t = t$ when $\{L_t^Y = 0\}$. Consequently

$$\lim_{t\to 0} \boldsymbol{E}^{(x,y)}\Big[\frac{g(X_t, Y_t) - g(x,y)}{t}\Big] = \frac{1}{2}\partial_y^2 g(x,y)\,.$$

For $y = 1$ we note

$$\lim_{t\to 0} \boldsymbol{E}^{(x,1)}\Big[\frac{g(X_t, Y_t) - g(x,y)}{t}\Big] \\ = \frac{1}{2}\partial_y^2 g(x,1) + \lim_{t\to 0} \boldsymbol{E}^{(x,1)}\Big[\frac{1}{t}\int_0^t \partial_y g(X_s, Y_s)\, dY_s\Big]\,. \tag{3.10}$$

By (3.5) we know $\boldsymbol{E}^{(x,1)} L_t^Y(1) = O(\sqrt{t})$, and hence the right hand side of (3.10) is finite if and only if $\partial_y g(x,1) = 0$.

Finally, we compute the generator on the spine $y = 0$. First we show that if we start Y at 0 then for a short time it spends "most" of the time at 0. More precisely we claim

$$\lim_{t\to 0} \boldsymbol{E}^0\Big[\frac{T_t}{t}\Big] = 0\,. \tag{3.11}$$

Let M_t be the running maximum of $\tilde{B}$. Note that since $L^{\bar{B}} = L^{\tilde{B}}$ on $\{M_t < 1\}$, we have

$$\boldsymbol{P}^0\Big(L_t^{\bar{B}} \leqslant \lambda\Big) \leqslant \boldsymbol{P}^0\Big(L_t^{\tilde{B}} \leqslant \lambda\Big) + \boldsymbol{P}^0\Big(M_t > 1\Big) \\ = 1 - 2\boldsymbol{P}^0\Big(\lambda < \tilde{B}_t < 1\Big) \leqslant \sqrt{\frac{2}{\pi}}\Big(\frac{\lambda}{\sqrt{t}} + \sqrt{t}e^{-\frac{1}{2t}}\Big)\,.$$

Thus,

$$\begin{aligned}\boldsymbol{E}\Big[\frac{T_t}{t}\Big] &= \int_0^1 \boldsymbol{P}\Big(T_t > st\Big)\, ds = \int_0^1 \boldsymbol{P}\Big(st + 2L_{st}^{\bar{B}} \leqslant t\Big)\, ds \\ &= \int_0^1 \boldsymbol{P}\Big(L_{st}^{\bar{B}} \leqslant \frac{(1-s)t}{2}\Big)\, ds \leqslant \int_0^1 \sqrt{\frac{2}{\pi}}\Big(\frac{2(1-s)}{\sqrt{s}}\sqrt{t} + \sqrt{st}\, e^{-\frac{1}{2st}}\Big)\, ds \\ &\leqslant C\sqrt{t}\,.\end{aligned}$$

With this estimate, we can now compute generator on the spine. Using equation (3.11) we see

$$\boldsymbol{E}^0\left[\frac{L_t^Y}{t}\right] = \boldsymbol{E}^0\left[\frac{L_{T_t}^{\bar{B}}}{t}\right] = \frac{1}{2}\boldsymbol{E}^0\left[\frac{t - T_t}{t}\right] \xrightarrow{t\to 0} \frac{1}{2}\,. \tag{3.12}$$

Using (3.5) we have,

$$\boldsymbol{E}^0\left[\frac{Y_t}{t}\right] = \boldsymbol{E}^0\left[\frac{\bar{B}_{T_t}}{t}\right] = \boldsymbol{E}^0\left[\frac{\tilde{B}_{T_t} + L_{T_t}^{\bar{B}}(0) - L_{T_t}^{\bar{B}}(1)}{t}\right]\,.$$

Since $T_t \leqslant t$, the third term tends to 0 and using the modulus of continuity for Brownian motion the first term does as well. Therefore we also have

$$\boldsymbol{E}^0\left[\frac{Y_t}{t}\right] \xrightarrow{t\to 0} \frac{1}{2}\,. \tag{3.13}$$

Thus using (3.11), (3.12) and (3.13) in equation (3.9) gives

$$\lim_{t\to 0}\frac{1}{t}\boldsymbol{E}^{(x,y)}\Big[g(X_t,Y_t)-g(x,y)\Big]=\frac{1}{2}\partial_y g(x,0)+\frac{1}{2}\partial_x^2 g(x,0)+0\,,$$

finishing the proof. □

Lemma 3.3. *Weak uniqueness holds for the martingale problem for A.*

The proof of Lemma 3.3 relies on the existence of regular solutions to the corresponding parabolic equation. We state this result next.

Lemma 3.4. *For all $f \in \mathcal{D}(A)$, there exists a solution to*

$$\partial_t u - Au = 0\,,\qquad u(\cdot,0)=f\,,\qquad \text{with } u(\cdot,t)\in\mathcal{D}(A)\,. \tag{3.14}$$

Given Lemma 3.4, the proof of Lemma 3.3 is standard (see for instance [RW00, EK86]). For the readers convenience, we describe it briefly here.

Proof of Lemma 3.3. Suppose Z, Z' are two processes satisfying the martingale problem for A. Let $f \in \mathcal{D}(A)$ be any test function, and u be the solution in $\mathcal{D}(A)$ of $\partial_t u - Au = 0$ with initial data f. Then for any $z \in \Omega_0$, $u(Z_t, T-t)$ and $u(Z'_t, T-t)$ are both martingales under the measure P^z. Hence

$$\begin{aligned}\boldsymbol{E}^\mu f(Z_T) &= \int_{\Omega_0}\boldsymbol{E}^z f(Z_T)\,\mu(dz) = \int_{\Omega_0}\boldsymbol{E}^z u(Z_t,T-t)\,\mu(dz) = \int_{\Omega_0} u(z,T)\,\mu(dz)\\ &= \int_{\Omega_0}\boldsymbol{E}^z u(Z'_t,T-t)\,\mu(dz) = \int_{\Omega_0}\boldsymbol{E}^z f(Z'_T)\,\mu(dz) = \boldsymbol{E}^\mu f(Z'_T)\,.\end{aligned}$$

Since $\mathcal{D}(A)$ is dense in $C_0(\Omega_0)$ this implies Z and Z' have the same one dimensional distributions. By the Markov property, this in turn implies that the laws of Z and Z' are the same. □

It remains to prove Lemma 3.4.

Proof of Lemma 3.4. Since u satisfies (3.14) we must have $\partial_t u = \frac{1}{2}\partial_y^2 u$ for $y \in (0,1)$. First we derive the identity

$$\partial_y u(x,0,t) = -\int_0^t K'_{t-s}(0,0)\partial_t u(x,0,s)\,ds + \int_0^1 \partial_y K_t(0,z) f(x,z)\,dz\,. \tag{3.15}$$

Here K' is the heat kernel on $(0,1)$ with Neumann boundary conditions at $y = 0$ and Dirichlet boundary conditions at $y = 1$, and K is the heat kernel on $(0,1)$ with Dirichlet boundary conditions at $y = 0$ and Neumann boundary conditions at $y = 1$.

We show (3.15) for each fixed x, so in the following we suppress the dependence on x. Suppose v satisfies the heat equation on $(0,1)$ with homogeneous Neumann boundary conditions at $y = 1$ and homogeneous initial data, i.e.

$$\begin{aligned} \partial_t v - \kappa\partial_y^2 v &= 0 &&\text{for } y\in(0,1), t>0,\\ v(0,t) &= g(t) &&\text{for } t>0,\\ \partial_y v(1,t) &= 0 &&\text{for } t>0,\\ \text{and}\qquad v(y,0) &= 0 &&\text{for } y\in(0,1).\end{aligned}$$

Proposition 3.5 (Dirichlet to Neumann Map)**.** *If v is as above, then*

$$\partial_y v(0,t) = -\int_0^t K'_{t-s}(0,0) g'(s)\,ds \tag{3.16}$$

Proof of Proposition 3.5. Let h be smooth on $(0,1)$ such that $h(1)=0$ and $h'(0)=1$ and set

$$w(y,t)=\partial_y v(y,t)-\frac{1}{\kappa}h(y)g'(t).$$

Then

$$\begin{aligned} \partial_t w-\kappa\partial_y^2 w &= -\frac{1}{\kappa}hg''+h''g' && \text{for } y\in(0,1),\ t>0,\\ \partial_y w(0,t)&=0 && \text{for } t>0,\\ w(1,t)&=0 && \text{for } t>0,\\ \text{and}\qquad w(y,0)&=-\frac{1}{\kappa}h(y)g'(0) && \text{for } y\in(0,1). \end{aligned}$$

By Duhamel's formula we know

$$w(y,t)=-\int_0^1 K_t'(y,z)\frac{1}{\kappa}h(z)g'(0)\,dz+\int_0^t\int_0^1 K_{t-s}'(y,z)\big(-\frac{1}{\kappa}hg''+h''g'\big)\,dz\,ds,$$

where K' is the heat kernel on $(0,1)$ with homogeneous Neumann boundary conditions at $y=0$ and Dirichlet boundary conditions at $y=1$. After integration by parts, we observe

$$\begin{aligned}\int_0^t\int_0^1 K_{t-s}'(y,z)\frac{1}{\kappa}h(z)g''(s)\,dz\,ds &= \frac{1}{\kappa}h(y)g'(t)-\int_0^1 K_t'(y,z)\frac{1}{\kappa}h(z)g'(0)\,dz\\ &\quad+\int_0^t\int_0^1 \partial_t K_{t-s}'(y,z)\frac{1}{\kappa}h(z)g'(s)\,dz\,ds.\end{aligned}$$

Since $\partial_t K'=\kappa\partial_z^2K'$, this implies

$$\begin{aligned} w(y,t)&=-\frac{1}{\kappa}h(y)g'(t)+\int_0^t g'(s)\int_0^1 (K_{t-s}'(y,z)h''(z)-\partial_z^2K_{t-s}'(y,z)h(z))\,dz\,ds\\ &=-\frac{1}{\kappa}h(y)g'(t)-\int_0^t g'(s)K_{t-s}'(y,0)\,ds\,. \end{aligned}$$

Consequently,

$$\partial_y v(y,t)=-\int_0^t K_{t-s}'(y,0)g'(s)\,ds.$$

□

Now suppose $\tilde v$ satisfies

$$\begin{aligned} \partial_t\tilde v-\kappa\partial_y^2\tilde v&=0 && \text{for } y\in(0,1),t>0,\\ \tilde v(0,t)&=0 && \text{for } t>0,\\ \partial_y\tilde v(1,t)&=0 && \text{for } t>0,\\ \text{and}\qquad \tilde v(y,0)&=f(y) && \text{for } y\in(0,1). \end{aligned}$$

Then

$$\tilde v(y,t)=\int_0^1 K_t(y,z)f(z)\,dz$$

and so

$$\partial_y\tilde v(0,t)=\int_0^1 \partial_y K_t(0,z)f(z)\,dz\,. \tag{3.17}$$

Hence (3.15) follows from (3.16), (3.17) and linearity. The flux condition (3.8) guarantees the following corollary,

Corollary 3.6. *Let $v(x,t) = u(x,0,t)$, where u is the solution of* (3.14). *Then v satisfies*

$$\partial_t v + \frac{1}{2}\partial_t^w v - \frac{1}{2}\partial_x^2 v = g\,, \tag{3.18}$$

where

$$g(x,t) \stackrel{\text{def}}{=} \frac{1}{2}\int_0^1 \partial_y K_t(0,z) f(x,z)\,dz\,, \quad \text{and} \quad \partial_t^w v(x,t) \stackrel{\text{def}}{=} \int_0^t w(t-s)\partial_t v(x,s)\,ds\,,$$

with kernel $w(t) = K_t'(0,0)$. Note, the operator ∂_t^w above is a generalized Caputo derivative.[1]

Existence for (3.18) follows from Lemma 2.6. Since u satisfies the heat equation for $y \in (0,1)$ we can write u in terms of v and f using the heat kernel. Explicitly, we have

$$u(x,y,t) = \int_0^1 K_t(y,z)f(z)\,dz\,, + \kappa\int_0^t \partial_z K_{t-s}(y,0)v(x,s)\,ds\,,$$

where K is the heat kernel on $(0,1)$ with Dirichlet boundary conditions at $y=0$ and Neumann boundary conditions at $y=1$. Since v is $C^{2,1}$ this immediately implies $u \in C^{2,1}$. Thus to show $u(\cdot,t) \in \mathcal{D}(A)$ we only need to verify the flux condition (3.8). This, however, follows immediately from the fact that $\partial_y^2 u(x,0,t) = 2\partial_t u(x,0,t) = 2\partial_t v(x,0,t)$ and equations (3.15) and (3.18). □

We can also characterize the kernel w by it's Laplace transform

Proposition 3.7. *The kernel w in* (3.18) *is a function whose Laplace transform is given by*

$$\mathcal{L}w(s) = \int_0^t e^{-st}w(t)\,dt = \frac{2\tanh(\sqrt{2s})}{\sqrt{2s}}\,. \tag{3.19}$$

Proof of Proposition 3.7. Let u solve (3.14) and let $g = g(x,y,t)$ be the solution to

$$\begin{aligned} \partial_t g - \frac{1}{2}\partial_y^2 g &= 0 && \text{for } t > 0,\ y \in (0,1)\,,\\ g(x,0,t) = g(x,1,t) &= 0 && \text{for } t > 0\,,\\ g(x,y,0) = f(x,y) &- f(x,0) && \text{for } y \in (0,1),\ t = 0\,. \end{aligned}$$

Define $u_1 = u - g$, and observe that u_1 satisfies the heat equation with initial data $u_1(x,y,0) = f(x,0) = v(x,0) = v_0(x)$, and boundary conditions

$$u_1(x,0,t) = u(x,0,t) = v(x,t) \qquad \text{and} \qquad \partial_y u_1(x,1,t) = 0\,.$$

Taking the Laplace transform yields the ODE in the variable y

$$sU_1 - v_0 - \frac{1}{2}\partial_y^2 U_1 = 0\,,$$

[1]The reason for this terminology is that if instead of K one used the Neumann heat kernel on the entire half line, then $w(t) = 1/\sqrt{\pi\kappa t}$. In this case $\sqrt{\kappa}\partial_t^w$ is precisely the Caputo fractional derivative of order 1/2 (see for instance [Die10]).

with initial condition $U_1(s) = V(s)$, and $\partial_y U_1(1) = 0$. Solving this ODE yields

$$U_1(x,y,s) = \frac{v_0}{s} + \Big(\frac{1}{1+e^{2\sqrt{2s}}}\Big)\Big(V - \frac{v_0}{s}\Big)\Big[e^{y\sqrt{2s}} + e^{\sqrt{2s}(2-y)}\Big]\,,$$

and hence

$$\partial_y U_1(x,0,s) = -\sqrt{2s}\Big(V - \frac{v_0}{s}\Big)\tanh\sqrt{2s} = -\frac{2\tanh\sqrt{2s}}{\sqrt{2s}}\Big(sV - v_0\Big)\,.$$

This implies $\partial_y u_1(x,0,t) = -\partial_t^w v(x,t)$, where w has the Laplace transform given by (3.19). Note here $\partial_y g(x,0,t)$ gives the forcing in equation (3.18). ☐

Remark 3.8. The direct calculation using the Laplace transform above obtains an expression for the Laplace transform of $\partial_y U_1$ at $y = 0$. To express $\partial_y u_1$ at $y = 0$ as the time convolution operator $\partial_t w$, one needs to prove the existence of a non-negative function w whose Laplace transform is given by the identity (3.19). The standard way to do this is to use Bernstein's theorem [Fel71, §XIII.4] and check that $\mathcal{L}w$ is a completely monotone function. Unfortunately, in our case, this condition is not easy to check.

3.2. **Abstract construction using the Hille-Yosida Theorem.** For completeness, we conclude this section with an abstract construction of the process Z (defined in Lemma 3.1) using the Hille-Yosida theorem. The idea is to start with the differential operator A from the comb model, and show that it generates a C^0- Markov semigroup on an appropriate L^2 space, which has an associated Markov process.

Theorem 3.9 (Hille-Yosida)**.** *A symmetric operator A on a Hilbert space H generates a C^0-Markov semigroup if and only if*

1. *A is closed and densely defined*
2. *A is non-positive, i.e. $\langle Ax, x\rangle_H \leqslant 0$ for all $x \in \mathcal{D}(A)$.*

where $\langle\cdot,\cdot\rangle$ is the inner product on H.

Remark 3.10. Note that item 2 above is sufficient to establish the resolvent estimates

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{1}{\lambda}$$

which can be found in the general statement of the Hille-Yosida theorem.

To accommodate the boundary condition which involves the second derivative, we need to pack the boundary value into the function space as well. Let $\Omega = \mathbb{R}\times(0,1)$ and

$$H = L^2(\Omega) \oplus L^2(\mathbb{R}, (1/\alpha)\lambda)$$

where $\alpha > 0$ and λ is the Lebesgue measure on $\mathbb{R}$ and let

$$\mathcal{D}(\tilde{A}) = \{(u, u(\cdot,0)) \mid u \in H^1(\Omega), \partial_y^2 u \in L^2(\Omega), \partial_y u(x,1) = 0, u(\cdot,0) \in H^2(\mathbb{R})\}\,.$$

For $(u, u(\cdot,0)) \in \mathcal{D}(\tilde{A})$ we define

$$\tilde{A}(u, u(\cdot,0)) = \big(\partial_y^2 u, \alpha\partial_y u(\cdot,0) + \partial_x^2 u(\cdot,0)\big)\,.$$

Lemma 3.11. *The operator $\tilde{A}$ above is non-positive, closed and densely defined.*

Proof. Let $x = (u, u(\cdot,0)) \in \mathcal{D}(\tilde{A})$ and observe after integration by parts,

$$\langle\tilde{A}x, x\rangle_H = \int_\Omega u\partial_y^2 u + \int_{\mathbb{R}} \frac{1}{\alpha}(\alpha\partial_y u(\cdot,0) + \partial_x^2 u(\cdot,0))u(\cdot,0) \tag{3.20}$$

$$= -\int_\Omega (\partial_y u)^2 - \frac{1}{\alpha}\int_{\mathbb{R}} (\partial_x u(\cdot,0))^2 \leqslant 0\,,$$

which shows $\tilde{A}$ is non-positive. Next we show $\tilde{A}$ is closed. Suppose $x_n = (u_n, u_n(\cdot,0)) \to (u,\hat{u})$ in H and $\tilde{A}x_n \to (f,b)$ in H. Using (3.20)

$$\int_\Omega (\partial_y u_n)^2 + \frac{1}{\alpha}\int_{\mathbb{R}} (\partial_x u_n(\cdot,0))^2 = -\int_\Omega u_n \partial_y^2 u_n - \int_{\mathbb{R}} u_n(\cdot,0)(\partial_y u_n(\cdot,0) + \frac{1}{\alpha}\partial_x^2 u_n(\cdot,0))$$

and so as the right side is bounded we see $\partial_y u_n$ is bounded in $L^2(\Omega)$ and hence up to a subsequence converges weakly to some $v \in L^2(\Omega)$. Similarly $\partial_x u_n(\cdot,0)$ converges in $L^2(\mathbb{R})$. Now let $\phi \in C_c^\infty(\Omega)$, multiply to $\partial_y^2 u_n$ and integrate to see

$$\int_\Omega \phi \partial_y^2 u_n = -\int_\Omega \partial_y \phi \partial_y u_n\,.$$

Taking limits on both sides we obtain

$$\int_\Omega \phi f = -\int_\Omega \partial_y \phi v\,,$$

and hence $\partial_y v = f \in L^2(\Omega)$. If we apply the same argument by testing ϕ against $\partial_y u_n$ we find $v = \partial_y u$ and so $f = \partial_y^2 u$. By the continuity of traces from H^1 to L^2, and since u_n, $\partial_y u_n$ and $\partial_y^2 u_n$ converge in L^2, we know $u_n(\cdot,0)$ and $\partial_y u_n(\cdot,0)$ converge to $u(\cdot,0)$ and $\partial_y u(\cdot,0)$ respectively in $L^2(\mathbb{R})$. The latter also tells us $\partial_x^2 u_n(\cdot,0)$ converges to $(1/\alpha)b - \partial_y u(\cdot,0)$ in $L^2(\mathbb{R})$. Now consider a test function $\psi \in C_c^\infty(\mathbb{R})$, and apply the same arguments as with ϕ but now testing against $\partial_x u(\cdot,0)$ and $\partial_x^2 u(\cdot,0)$ to establish $u(\cdot,0) \in H^2(\mathbb{R})$ with $\partial_x u_n(\cdot,0) \to \partial_x u(\cdot,0)$ and $\partial_x^2 u_n(\cdot,0) \to \partial_x^2 u(\cdot,0)$ in $L^2(\mathbb{R})$. Combining all these shows us that $\tilde{A}x_n \to \tilde{A}(u,u(\cdot,0))$ which shows $\tilde{A}$ is closed. For density, let $\varepsilon > 0$ and $(u,v) \in H$. There exists $\psi \in C_c^\infty(\Omega)$ such that $\|u-\psi\|_{L^2(\Omega)} < \varepsilon$ and $\phi \in C_c^\infty(\mathbb{R})$ such that $\|v-\phi\|_{L^2(\mathbb{R})} < \varepsilon$. Next let $h \in C^\infty([0,1];\mathbb{R}^+)$ be any positive function such that $h(0) = 1$ and $\|h\|_{L^2((0,1))} < \varepsilon/\|\phi\|_{L^2(\mathbb{R})}$. Now we define $w : \Omega \to \mathbb{R}$ by $w(x,y) = \psi(x,y) + \phi(x)h(y)$. By construction we have $(w, w(\cdot,0)) \in \mathcal{D}(\tilde{A})$ and $w(x,0) = \phi(x)$. Therefore we can see

$$\begin{aligned}\|(u,v) - (w,w(\cdot,0))\|_H &= \|w-u\|_{L^2(\Omega)} + \|\phi - v\|_{L^2(\mathbb{R})}\\ &\leqslant \|\psi - u\|_{L^2(\Omega)} + \|h(x)\phi(y)\|_{L^2(\Omega)} + \|\phi - v\|_{L^2(\mathbb{R})}\\ &\leqslant 3\varepsilon\end{aligned}$$

which shows that $\mathcal{D}(\tilde{A})$ is dense in H. □

We still have not clarified the relation between $\tilde{A}$ and the operator A in §4 which generates the comb model. By the Hille-Yosida theorem and Lemma 3.11, $\tilde{A}$ generates a C^0-Semigroup $T(t)$ on H, so let $f \in H$ and since $T : H \to \mathcal{D}(A)$ we can write $T(t)f = (u(x,y,t), u(x,0,t))$. Then u satisfies $\partial_t u = \partial_y^2 u$, $u(x,y,0) = f_1(x,y)$ and $Au \in \mathcal{D}(A)$ as well and hence $\partial_y^2 u(x,0,t) = \alpha \partial_y u(x,0,t) + \partial_x^2 u(x,0,t)$ which is precisely the boundary condition in the limiting comb model (3.8).

4. **Thin Comb Model: The SDE approach**

The aim of this section is to study the process Z^ε, the diffusion associated with (1.10)–(1.13), in terms of the underlying SDE. Roughly speaking the process Z^ε is a "skew" Brownian that enters the teeth with small probability $(\varepsilon/(2+\varepsilon))$ at the junction points $\varepsilon\mathbb{Z} \times \{0\}$, and continues in the spine otherwise. Using martingale methods we will show that as $\varepsilon \to 0$ the processes Z^ε converge to a diffusion with a *sticky* reflection at the spine.

4.1. **The SDE description of Z^ε.** Let W_t be a standard Brownian motion on $\mathbb{R}$. Consider the system

$$dX_t^\varepsilon = \mathbf{1}_{\{Y_t^\varepsilon = 0\}}\, dW_t\,, \tag{4.1}$$

$$dY_t^\varepsilon = \mathbf{1}_{\{Y_t^\varepsilon > 0\}}\, dW_t + \frac{\varepsilon}{2+\varepsilon}\, d\ell_t - dL_t^{Y^\varepsilon}(1) \tag{4.2}$$

where $L_t^{Y^\varepsilon}(1)$ is the local time of the process Y^ε about 1, and

$$\ell_t = L^{Z^\varepsilon}(\varepsilon\mathbb{Z} \times \{0\})$$

is the local time of the joint process $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ about the junction points $\varepsilon\mathbb{Z} \times \{0\}$. Explicitly,

$$\ell_t = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t \mathbf{1}_{\{d(Z^\varepsilon(s), \varepsilon\mathbb{Z}) < \delta\}}\, ds\,, \tag{4.3}$$

where d denotes the graph distance between two points in Ω_ε. The process Z^ε can be viewed like a Brownian motion on Ω_ε which upon hitting a vertex in $\varepsilon\mathbb{Z} \times \{0\}$ enters the vertical teeth with probability $\frac{\varepsilon}{(2+\varepsilon)}$. As we will shortly see, the process Z^ε is precisely the diffusion associated with (1.10)–(1.13).

A weak solution to the system (4.1)–(4.2) can be constructed abstractly as follows. Define the linear operator $\mathcal{L}^\varepsilon$ by

$$\mathcal{L}^\varepsilon f = \begin{cases} \frac{1}{2}\partial_y^2 f & \text{if } (x,y) \in \varepsilon\mathbb{Z} \times (0,1)\,, \\ \frac{1}{2}\partial_x^2 f & \text{if } (x,y) \in \mathbb{R} \times \{0\}\,. \end{cases}$$

Now define the domain $\mathcal{D}(\mathcal{L}^\varepsilon)$ to be the set of all functions

$$f \in C_0(\Omega_\varepsilon) \cap C_b^2(\Omega_\varepsilon - (\varepsilon\mathbb{Z} \times \{0\}))$$

such that $\mathcal{L}^\varepsilon f \in C_0(\Omega_\varepsilon)$ and

$$\begin{aligned} \frac{\varepsilon}{2}\partial_y f(x,0) + \frac{1}{2}\partial_x^+ f(x,0) - \frac{1}{2}\partial_x^- f(x,0) &= 0 \quad \text{for } x \in \varepsilon\mathbb{Z}\,, \\ \partial_y f(x,1) &= 0 \quad \text{for } x \in \varepsilon\mathbb{Z}\,. \end{aligned}$$

Theorem 2.1 can be used to show the existence of a continuous Fellerian Markov process $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ that has generator $\mathcal{L}^\varepsilon$. By choice of $\mathcal{L}^\varepsilon$ and $\mathcal{D}(\mathcal{L}^\varepsilon)$ the process Z^ε is clearly the diffusion associated with (1.10)–(1.13). We claim that Z^ε is also the unique weak solution to the SDE (4.1)–(4.2).

Proposition 4.1. *The process $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ is a weak solution to the system (4.1)–(4.2) with initial distribution μ^ε. That is, there is a non-decreasing processes ℓ_t adapted to $\mathcal{F}_t^{Z^\varepsilon}$ and a standard Brownian motion W_t such that (4.1)–(4.2) holds. Moreover, under the additional assumption that process spends measure zero time at the junctions, i.e.*

$$\int_0^t \mathbf{1}_{\{Y_s^\varepsilon = 0, X_s^\varepsilon \in \varepsilon\mathbb{Z}\}}\, ds = 0\,,$$

the solutions are unique in law.

Proof of Proposition 4.1. Equations (4.1) and (4.2) follow from Lemma 2.2 by choosing $F(x, y) = x$ and $F(x, y) = y$ respectively. The proof of weak uniqueness is similar to the proof in [EP14] and we do not present it here. □

4.2. **Convergence as $\varepsilon \to 0$.** The main result of this section identifies the limit of Z^ε as $\varepsilon \to 0$.

Theorem 4.2. *Viewing the initial distributions μ^ε as probability measures on $\Omega_\varepsilon \subseteq \Omega_0$, suppose that $(\mu^\varepsilon) \to \mu$ weakly as $\varepsilon \to 0$. Then the processes Z^ε converge weakly (as $\varepsilon \to 0$) to the process $Z \overset{\text{def}}{=} (X, Y)$ which is a weak solution to the system (3.2)-(3.4) with initial distribution μ.*

Note for any $\varepsilon > 0$, the process Z^ε behaves like a skew Brownian motion at the junction points $\varepsilon\mathbb{Z} \times \{0\}$. However, as $\varepsilon \to 0$, the limiting process develops a *sticky* reflection on the spine $y = 0$. Before we prove Theorem 4.2, we need a few lemmas.

Lemma 4.3. *Let $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ be the process on the thin comb Ω_ε, as defined above. Then for any $T > 0$, the family of processes Z^ε is tight on $C([0, T]; \mathbb{R}^2)$.*

Proof of Lemma 4.3. We write both X^ε and Y^ε as time changed Brownian motions as follows. Let $S(t) = \int_0^t \mathbf{1}_{\{Y_s^\varepsilon = 0\}} \, ds$. Then letting $S^{-1}(t)$ be the right-continuous inverse, by the Dambis-Dubins-Schwartz time change theorem, $\bar{W}_t = X^\varepsilon_{S^{-1}(t)}$ is a Brownian motion and $X_t^\varepsilon = \bar{W}_{S(t)}$. Similarly we can time change Y^ε using $R(t) = \int_0^t \mathbf{1}_{\{Y_t^\varepsilon > 0\}} \, ds$. Equation (4.2) tells us that $\bar{B}_t = Y^\varepsilon_{R^{-1}(t)}$ satisfies

$$d\bar{B}_t = d\tilde{B}_t + dL_t^{\bar{B}}(0) - dL_t^{\bar{B}}(1)\,.$$

where $\tilde{B}_t$ is a Brownian motion and hence $\bar{B}_t$ is a doubly-reflected Brownian motion on $[0, 1]$ such that $Y_t^\varepsilon = \bar{B}_{R(t)}$. Since $S(t) - S(s) \leqslant t - s$ and $R(t) - R(s) \leqslant t - s$ holds with probability one, the moduli of continuity of X^ε and Y^ε over $[0, T]$ are no more than those of $\bar{W}$ and $\bar{B}$ over $[0, T]$, respectively. This implies tightness. □

Lemma 4.4. *Let A be the generator (3.7). If $f \in \mathcal{D}(A)$, and $K \subseteq \Omega_0$ is compact as a subset of $\mathbb{R}^2$ with the usual topology, then*

$$\lim_{\varepsilon \to 0} \sup_{z \in K \cap \Omega_\varepsilon} \boldsymbol{E}^z \Big(f(Z_t^\varepsilon) - f(Z_0) - \int_0^t Af(Z_s^\varepsilon) \, ds \Big) = 0$$

Proof of Lemma 4.4. We claim for any $k \in \mathbb{N}$ we have

$$L^{Z^\varepsilon}(\varepsilon k, 0) = L^{X^\varepsilon}(\varepsilon k, 0) + L^{Y^\varepsilon}(\varepsilon k, 0)\,, \quad \text{and} \quad L^{Y^\varepsilon}(\varepsilon k, 0) = \frac{\varepsilon}{2} L^{X^\varepsilon}(\varepsilon k, 0)\,.$$

which are consequences of equation (2.3). (The second equality can also be deduced the independent excursion construction in §5, below). Consequently

$$L^{Z^\varepsilon}(\varepsilon k, 0) = \frac{2 + \varepsilon}{2} L^{X^\varepsilon}(\varepsilon k, 0) = \frac{2 + \varepsilon}{\varepsilon} L^{Y^\varepsilon}(\varepsilon k, 0)\,. \tag{4.4}$$

For any $f \in \mathcal{D}(A)$, Lemma 2.2 gives

$$\begin{aligned} f(Z_t^\varepsilon) - f(Z_0^\varepsilon) = \int_0^t \partial_y f(Z_s^\varepsilon) \mathbf{1}_{\{Y_s^\varepsilon > 0\}} \, dY_s^\varepsilon + \int_0^t \partial_x f(Z_s^\varepsilon) \mathbf{1}_{\{Y_s^\varepsilon = 0\}} \, dX_s^\varepsilon \\ + \int_0^t \frac{1}{2} \partial_y^2 f(Z_s^\varepsilon) \mathbf{1}_{\{Y_s^\varepsilon > 0\}} + \frac{1}{2} \partial_x^2 f(Z_s^\varepsilon) \mathbf{1}_{\{Y_s^\varepsilon = 0\}} \, ds \end{aligned} \tag{4.5}$$

$$+\sum_{k\in\mathbb{Z}}\Big(\frac{\varepsilon}{2+\varepsilon}\partial_y f(\varepsilon k,0)+\frac{1}{2+\varepsilon}\big(\partial_x^+ f(\varepsilon k,0)-\partial_x^- f(\varepsilon k,0)\big)\Big)L_t^{Z^\varepsilon}(\varepsilon k,0)\,.$$

The first integral on the right of equation (4.5) can be rewritten as

$$\begin{aligned}\int_0^t \partial_y f(Z_s^\varepsilon)\mathbf{1}_{\{Y_s^\varepsilon>0\}}\,dY_s^\varepsilon &= \int_0^t \partial_y f(Z_s^\varepsilon)\mathbf{1}_{\{Y_s^\varepsilon>0\}}\,dW_s - \int_0^t \partial_y f(X_s^\varepsilon,1)\,dL_s^{Y^\varepsilon}(1)\\ &= \int_0^t \partial_y f(Z_s^\varepsilon)\mathbf{1}_{\{Y_s^\varepsilon>0\}}\,dW_s\,.\end{aligned}$$

Here we used the fact that $\partial_y f(x,1)=0$ for any $f\in\mathcal{D}(A)$.

Returning to (4.5), we note that $f\in C^2(\mathbb{R}\times\{0\})$ implies $\partial_x^+ f(\varepsilon k,0)=\partial_x^- f(\varepsilon k,0)$. Thus for $(x,y)\in K\cap\Omega_\varepsilon$, taking expectations on both sides and using (4.4) gives

$$\begin{aligned}&\boldsymbol{E}^{(x,y)}\Big(f(Z_t^\varepsilon)-f(Z_0^\varepsilon)-\int_0^t Af(Z_s^\varepsilon)\,ds\Big)\\ &\quad=\frac{1}{2}\boldsymbol{E}^{(x,y)}\Big(\int_0^t \partial_y^2 f(Z_s^\varepsilon)\mathbf{1}_{\{Y_s^\varepsilon>0\}}+\partial_x^2 f(Z_s^\varepsilon)\mathbf{1}_{\{Y_s^\varepsilon=0\}}-\partial_y^2 f(Z_s^\varepsilon)\,ds\\ &\qquad\qquad+\varepsilon\sum_{k\in\mathbb{Z}}\partial_y f(\varepsilon k,0)L_t^{X^\varepsilon}(\varepsilon k,0)\Big)\\ &\quad=\frac{1}{2}\boldsymbol{E}^{(x,y)}\Big(-\int_0^t \partial_y f(X_s^\varepsilon,0)\mathbf{1}_{\{Y_s^\varepsilon=0\}}\,ds+\varepsilon\sum_{k\in\mathbb{Z}}\partial_y f(\varepsilon k,0)L_t^{X^\varepsilon}(\varepsilon k,0)\Big)\\ &\quad=I+II\,,\end{aligned}$$

where

$$\begin{aligned}I &\stackrel{\text{def}}{=}\frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{E}^{(x,y)}\int_0^t\big(\partial_y f(\varepsilon k,0)-\partial_y f(X_s^\varepsilon,0)\big)\mathbf{1}_{\{Y_s^\varepsilon=0,\ |X_s^\varepsilon-\varepsilon k|<\frac{\varepsilon}{2}\}}\,ds\,,\\ II &\stackrel{\text{def}}{=}\frac{1}{2}\sum_{k\in\mathbb{Z}}\partial_y f(\varepsilon k,0)\boldsymbol{E}^{(x,y)}\Big(\varepsilon L_t^{X^\varepsilon}-\int_0^t\mathbf{1}_{\{Y_s^\varepsilon=0,\ |X_s^\varepsilon-\varepsilon k|<\frac{\varepsilon}{2}\}}\,ds\Big)\,.\end{aligned}$$

Note that there exists Brownian motion W such that $X_t^\varepsilon=W_{S(t)}$ where $S(t)$, defined by

$$S(t)\stackrel{\text{def}}{=}\int_0^t\mathbf{1}_{\{Y^\varepsilon(s)=0\}}\,ds\,,\tag{4.6}$$

is the amount of time the joint process spends on the spine of the comb up to time t. To estimate I, for any $\delta>0$ we choose sufficiently large closed interval $C=[\varepsilon(c_0-1/2),\varepsilon(c_1+1/2)]\subset\mathbb{R}$ such that

$$\sup_{(x,y)\in K}\boldsymbol{E}^x\Big(\int_0^t\mathbf{1}_{\{W_s\notin C\}}\,ds\Big)<\frac{\delta}{\|\partial_y f\|_\infty}\,.\tag{4.7}$$

Then since $S(s)\leqslant s$, it follows that

$$\boldsymbol{P}^x(X_s^\varepsilon\notin C)\leqslant\boldsymbol{P}^x(W_s\notin C)$$

and so the above estimate can be applied for X^ε independent of ε. Since $\partial_y f(\cdot,0)$ is continuous and hence uniformly continuous on C, for any $\delta>0$ we can choose $\varepsilon>0$ such that if $x_1,x_2\in C$ with $|x_1-x_2|<\varepsilon$ then $|\partial_y f(x_1,0)-\partial_y f(x_2,0)|<\delta$. For such ε and for $k\in[c_0,c_1]$,

$$(4.8)\quad \boldsymbol{E}^{(x,y)} \int_0^t |\partial_y f(\varepsilon k, 0) - \partial_y f(X_s^\varepsilon, 0)| \mathbf{1}_{\{Y_s^\varepsilon = 0,\ |X_s^\varepsilon - \varepsilon k| < \frac{\varepsilon}{2}\}} \, ds \leqslant \delta \int_0^t \boldsymbol{P}^x \Big(|X_s^\varepsilon - \varepsilon k| < \frac{\varepsilon}{2} \Big) \, ds \,.$$

Combining the above with (4.7), gives the following estimate of I

$$|I| \leqslant \frac{1}{2} \Bigg(\delta \sum_{\varepsilon k \in C} \int_0^t \boldsymbol{P}^x \Big(|X_s^\varepsilon - \varepsilon k| < \frac{\varepsilon}{2} \Big) \, ds + 2 \|\partial_y f\|_\infty \sum_{\varepsilon k \notin C} \int_0^t \boldsymbol{P}^x \Big(|X_s^\varepsilon - \varepsilon k| < \frac{\varepsilon}{2} \Big) \, ds \Bigg) \leqslant \frac{t\delta}{2} + \delta.$$

Since $\delta > 0$ was arbitrary this proves $I \to 0$ as $\varepsilon \to 0$.

In order to estimate II, we again use the above representation to see

$$(4.9)\quad \boldsymbol{E}^{(x,y)} \Big| \varepsilon L_t^{X^\varepsilon}(\varepsilon k, 0) - \int_0^t \mathbf{1}_{\{Y_s^\varepsilon = 0,\ |X_s^\varepsilon - \varepsilon k| < \frac{\varepsilon}{2}\}} \, ds \Big| = \boldsymbol{E}^x \Big| \varepsilon L_{S(t)}^W(\varepsilon k) - \int_0^{S(t)} \mathbf{1}_{\{|W_s - \varepsilon k| < \frac{\varepsilon}{2}\}} \, ds \Big| \,,$$

where $S(t)$, is given by equation (4.6) above. Thus to show $II \to 0$, it suffices to estimate the right hand side of (4.9) as $\varepsilon \to 0$. Also, by shifting the indices of the sum to compensate, we can assume that $x = 0$.

To this end, let f_ε be defined by

$$f_\varepsilon(x) \stackrel{\text{def}}{=} \begin{cases} \varepsilon(\varepsilon k - x) - \dfrac{\varepsilon^2}{4} & \text{if } x < \varepsilon k - \dfrac{\varepsilon}{2} \,, \\ (x - \varepsilon k)^2 & \text{if } \varepsilon k - \dfrac{\varepsilon}{2} \leqslant x \leqslant \varepsilon k + \dfrac{\varepsilon}{2} \,, \\ \varepsilon(x - \varepsilon k) - \dfrac{\varepsilon^2}{4} & \text{if } x > \varepsilon k + \dfrac{\varepsilon}{2} \,. \end{cases}$$

By Ito's formula we have,

$$f_\varepsilon(W_t) - \varepsilon|W_t - \varepsilon k| - (f_\varepsilon(W_0) - \varepsilon|W_0 - \varepsilon k|) = \int_0^t (f_\varepsilon'(W_s) - \varepsilon \operatorname{sign}(W_s - \varepsilon k)) \, dW_s + \int_0^t \mathbf{1}_{\{|W_s - \varepsilon k| < \frac{\varepsilon}{2}\}} \, ds - \varepsilon L_t^W(\varepsilon k) \,.$$

Using the Itô isometry and the inequalities

$$\big| f_\varepsilon(x) - \varepsilon|x - \varepsilon k| \big| \leqslant \frac{\varepsilon^2}{4} \,,$$
$$|f_\varepsilon'(x) - \varepsilon \operatorname{sign}(x - \varepsilon k)| \leqslant \varepsilon \mathbf{1}_{[\varepsilon k - \frac{\varepsilon}{2}, \varepsilon k + \frac{\varepsilon}{2}]} \,,$$

we obtain

$$\boldsymbol{E}^0 \Big| \varepsilon L_t^W(\varepsilon k) - \int_0^t \mathbf{1}_{\{|W_s - \varepsilon k| < \frac{\varepsilon}{2}\}} \, ds \Big| \leqslant \frac{\varepsilon^2}{4} + \varepsilon \Big(\boldsymbol{E}^0 \int_0^t \mathbf{1}_{\{|W_s - \varepsilon k| < \frac{\varepsilon}{2}\}} \, ds \Big)^{\frac{1}{2}} \leqslant c(t) \varepsilon^{\frac{3}{2}} \,,$$

since

$$\boldsymbol{E}^0 \int_0^t \mathbf{1}_{\{|W_s - \varepsilon k| < \frac{\varepsilon}{2}\}} \, ds = \int_0^t \boldsymbol{P}^0 \Big(|W_s - \varepsilon k| < \frac{\varepsilon}{2} \Big) \, ds \leqslant c \int_0^t \frac{\varepsilon}{\sqrt{s}} \, ds = 2c\varepsilon\sqrt{t} \,.$$

We break up the sum in II and estimate as follows,

$$\frac{1}{2}\sum_{|k|\leqslant \frac{N}{\varepsilon}} |\partial_y f(\varepsilon k,0)| \boldsymbol{E}^0 \Big| \varepsilon L_t^{X^\varepsilon} - \int_0^t \mathbf{1}_{\{Y_s^\varepsilon=0,\ |X_s^\varepsilon-\varepsilon k|<\frac{\varepsilon}{2}\}}\, ds \Big| \leqslant \|\partial_y f\|_\infty \frac{2N}{\varepsilon} c(t)\varepsilon^{\frac{3}{2}}$$

and also

$$\frac{1}{2}\sum_{|k|> \frac{N}{\varepsilon}} |\partial_y f(\varepsilon k,0)| \boldsymbol{E}^0 \Big| \varepsilon L_t^{X^\varepsilon} - \int_0^t \mathbf{1}_{\{Y_s^\varepsilon=0,\ |X_s^\varepsilon-\varepsilon k|<\frac{\varepsilon}{2}\}}\, ds \Big|$$

$$\leqslant \frac{1}{2}\|\partial_y f\|_\infty \sum_{|k|>\frac{N}{\varepsilon}} \boldsymbol{E}^0 \Big(\varepsilon L_t^{X^\varepsilon} + \int_0^t \mathbf{1}_{\{Y_s^\varepsilon=0,\ |X_s^\varepsilon-\varepsilon k|<\frac{\varepsilon}{2}\}}\, ds \Big).$$

Combining, we have

$$|II| \leqslant \|\partial_y f\|_\infty \Big(\sum_{|k|>N/\varepsilon} \boldsymbol{E}^0[\varepsilon L_t^{X^\varepsilon}(\varepsilon k,0)] + \int_0^t \boldsymbol{P}^0\big(|X_s^\varepsilon| > N - \frac{\varepsilon}{2}\big)\, ds + 2Nc(t)\varepsilon^{\frac{1}{2}} \Big).$$

We can again use that X^ε has the same distribution as a Brownian motion with a time change $S(t) \leqslant t$ to replace X^ε with W, i.e.

$$|II| \leqslant \|\partial_y f\|_\infty \Big(\sum_{|k|>N/\varepsilon} \boldsymbol{E}^0[\varepsilon L_t^{W}(\varepsilon k)] + \int_0^t \boldsymbol{P}^0\big(|W_s| > N - \frac{\varepsilon}{2}\big)\, ds + Nc(t)\varepsilon^{\frac{1}{2}} \Big).$$

Notice that for fixed N, the first term converges as $\varepsilon \to 0$

$$\sum_{|k|>N/\varepsilon} \boldsymbol{E}^0[\varepsilon L_t^{W}(\varepsilon k)] \to \boldsymbol{E}^0 \left[\int_{|x|>N} L_t^W(x)\, dx \right].$$

Therefore, setting N sufficiently large and then sending $\varepsilon \to 0$ gives us $II \to 0$ as $\varepsilon \to 0$. This completes the proof. □

Proof of Theorem 4.2. Suppose first $Z^\varepsilon \to Z'$ weakly along some subsequence. We claim Z' should be a solution of the martingale problem for A with initial distribution μ. To see this set

$$M_t^\varepsilon = f(Z_t^\varepsilon) - f(Z_0^\varepsilon) - \int_0^t Af(Z_r^\varepsilon)\, dr$$

and observe

$$\boldsymbol{E}^{\mu^\varepsilon}\big(M_t^\varepsilon \bigm| \mathcal{F}_s\big) = M_s^\varepsilon + \boldsymbol{E}^{Z_s^\varepsilon}(M_{t-s}^\varepsilon),$$

by the Markov property. Using Lemmas 4.4 and 4.3, and taking limits along this subsequence, the last term on the right vanishes. Since this holds for all $f \in \mathcal{D}(A)$ and $\mathcal{D}(A)$ is dense in $C_0(\Omega_0)$, Z' must be a solution of the martingale problem for A. Since $Z^\varepsilon \to Z'$ weakly and $\mu^\varepsilon \to \mu$ weakly by assumption, we have $Z(0) \sim \mu$. By uniqueness of solutions to the martingale problem for A (Lemma 3.3), the above argument shows uniqueness of subsequential limits of $\mathbb{Z}^\varepsilon$. Combined with tightness (Lemma 4.3), this gives weak convergence as desired. □

5. Thin Comb Model: Excursion description.

In this section we use Itô's excursion theory (c.f. [Itô72,PY07]) to describe Z^ε, the diffusion associated with (1.10)–(1.13). We then identify Z^ε as a trapped Brownian motion in the framework of Ben Arous et. al. [BAC+15] and use this to provide an alternate description of the limiting behavior as $\varepsilon \to 0$.

5.1. The excursion decomposition of Z^ε.

The trajectories of Z^ε can be decomposed as a sequence of excursions from the junction points $\varepsilon\mathbb{Z} \times \{0\}$ into the teeth and spine respectively. The excursions into the teeth of the comb (excursions of Y^ε into $(0,1]$ while $X^\varepsilon \in \varepsilon\mathbb{Z}$) should be those of a reflected Brownian motion on $[0,1]$. The excursions into the spine (excursions of X^ε into $\mathbb{R} \setminus \varepsilon\mathbb{Z}$ while $Y^\varepsilon = 0$) should be be those of a standard Brownian motion on $\mathbb{R}$ between the points $\varepsilon\mathbb{Z}$. Thus one expects that that by starting with a standard Brownian motion $\bar{X}$ on $\mathbb{R}$ and an independent reflected Brownian motion $\bar{Y}$ on $[0,1]$, we can glue excursions of $\bar{X}$ and $\bar{Y}$ appropriately and obtain the diffusion Z^ε associated with (1.10)–(1.13). We describe this precisely as follows.

Let $\bar{X}$ be a standard Brownian motion on $\mathbb{R}$ and let $L_t^{\bar{X}}(x)$ denote its local time at $x \in \mathbb{R}$. Let $L_t^{\bar{X}}(\varepsilon\mathbb{Z})$, defined by

$$L_t^{\bar{X}}(\varepsilon\mathbb{Z}) \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} L_t^{\bar{X}}(\varepsilon k) = \lim_{\delta\to 0} \frac{1}{2\delta} \int_0^t \sum_{k\in\mathbb{Z}} \mathbf{1}_{(\varepsilon k-\delta, \varepsilon k+\delta)}(\bar{X}_s)\, ds\,,$$

denote the local time of $\bar{X}$ at the junction points $\varepsilon\mathbb{Z}$. Let $\tau^{\bar{X},\varepsilon}$ be the right-continuous inverse of $L_t^{\bar{X}}(\varepsilon\mathbb{Z})$ defined by

$$\tau^{\bar{X},\varepsilon}(\ell) = \inf\{t > 0 \mid L_t^{\bar{X}}(\varepsilon\mathbb{Z}) > \ell\}, \quad \ell \geqslant 0.$$

Notice that the functions $t \mapsto L_t^{\bar{X}}$ and $\ell \mapsto \tau^{\bar{X},\varepsilon}(\ell)$ are both non-decreasing.

Let $\bar{Y}$ be a reflected Brownian motion on $[0,1]$ which is independent of $\bar{X}$. As above, let $L^{\bar{Y}}(0)$ be the local time of $\bar{Y}$ about 0, and let $\tau^{\bar{Y}}$, defined by

$$\tau^{\bar{Y}}(\ell) = \inf\left\{t > 0 \mid L_t^{\bar{Y}}(0) > \ell\right\}\,,$$

be its right-continuous inverse. We define the random time-changes $\psi^{\bar{X},\varepsilon}$ and $\psi^{\bar{Y},\varepsilon}$ by

$$\psi^{\bar{X},\varepsilon}(t) = \inf\Big\{s > 0 \;\Big|\; s + \tau^{\bar{Y}}\Big(\frac{\varepsilon}{2} L_s^{\bar{X}}(\varepsilon\mathbb{Z})\Big) > t\Big\}\,, \tag{5.1}$$

and

$$\psi^{\bar{Y},\varepsilon}(t) = \inf\Big\{s > 0 \;\Big|\; s + \tau^{\bar{X},\varepsilon}\Big(\frac{2}{\varepsilon} L_s^{\bar{Y}}(0)\Big) > t\Big\}\,. \tag{5.2}$$

Note both $\psi^{\bar{X},\varepsilon}$ and $\psi^{\bar{Y},\varepsilon}$ are continuous and non-decreasing functions of time. The idea of (5.1) and (5.2) is to ensure that the local time balance (4.4) is satisfied which we prove below in Lemma 5.9.

Proposition 5.1. *The time changed process Z^ε defined by*

$$Z^\varepsilon(t) \stackrel{\text{def}}{=} \big(\bar{X}(\psi^{\bar{X},\varepsilon}(t)), \bar{Y}(\psi^{\bar{Y},\varepsilon}(t))\big)$$

has generator $\mathcal{L}^\varepsilon$ and is a weak solution to the system (4.1)–(4.2).

This gives an alternate and natural representation of $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$. For clarity of presentation, we postpone the proof of Proposition 5.1 to the end of this section (on page 37).

5.2. **Description as a trapped Brownian motion.** We now show how this representation can be explained in the framework of trapped Brownian motions as defined by Ben Arous, et.al. [BAC+15] (see §2.4). In this paper, the authors introduced a general framework to describe trapped random walks and Brownain motions and characterized all possible scaling limits of a fixed (random) trap structure. We shall see that our limiting model does not directly fit into this characterization because we not only scale the traps, but also the probability of entering the traps which is reflected through the flux balance condition. Consider a trapped Brownian motion $X_t = B(\psi(t))$ and then scale it to obtain the process

$$X^\varepsilon_t = \varepsilon X_{\rho(\varepsilon)^{-1} t}\,,$$

where ρ is a nondecreasing function. Then [BAC+15][Theorem 2.8] says if X^ε_t converges in law to some process $U(t)$ then U is either a Brownian motion, or an FK-SSBM mixtures which we now recall. Let $\mathcal{F}^\star$ be the set of all Laplace exponents of subordinators equipped with the topology of pointwise convergence. Such functions are called Bernstein functions and a characterization of this topology can be found in the appendix of [ILP15]. Let $\mathbb{F}$ be a σ finite measure on $\mathcal{F}^\star$ and (x_i, f_i) be a Poisson point process on $\mathbb{R}\times\mathcal{F}^\star$ with intensity measure $dx \times \mathbb{F}$. For each i, let S^i be an independent subordinator (from each other and from B) with Laplace exponent f_i. Define

$$\phi_t = \sum_i S^i_{L^B_t(x_i)} + V^\gamma_t$$

where V^γ is a γ-stable subordinator for some $\gamma \in (0,1)$. Then an FK-SSBM mixture is a process $U(t) = B(\psi(t))$ where

$$\psi(t) = \inf\bigl\{ s > 0 \bigm| \phi_s > t \bigr\}\,,$$

where ϕ is a subordinator as above.

As we saw in Theorem 1.2, the limiting process for the comb model is again a time changed Brownian motion as above but with

$$\phi_t = t + S_t$$

and S_t need not be a stable subordinator. See (3.12) for computation of the drift. The reason we do not necessarily obtain a stable law is because we are not scaling the time spent in the traps, but the probability of entering the trap. In fact by adopting different fixed geometries for the traps, we can obtain a variety of subordinators which arise as the inverse local time of some recurrent graph diffusion which is sticky at a point.

First, we identify the trap measure for X^ε. The process $\tau^{\bar{Y}}_\ell$, appearing in the time change (5.1), is a Lévy subordinator. Specifically, there is a Poisson random measure $N^{\bar{Y}}$ on $[0,\infty)\times[0,\infty)$ with intensity measure $d\ell \times \eta^{\bar{Y}}(s)ds$, where $\eta^{\bar{Y}}(s) : (0,\infty) \to (0,\infty)$, and such that

$$\tau^{\bar{Y}}_\ell = \int_{[0,\ell]}\int_{[0,\infty)} s N^{\bar{Y}}(d\ell \times ds). \tag{5.3}$$

In the definition of $\psi^{\bar{X},\varepsilon}(t)$ above, we have

$$\tau^{\bar{Y}}\Big(\frac{\varepsilon}{2}L_s^{\bar{X}}(\varepsilon\mathbb{Z})\Big) = \tau^{\bar{Y}}\Big(\sum_{k\in\mathbb{Z}}\frac{\varepsilon}{2}L_s^{\bar{X}}(\varepsilon k)\Big).$$

Because $\tau_\ell^{\bar{Y}}$ has stationary, independent increments, this is equal in law to

$$\tau^{\bar{Y}}\Big(\frac{\varepsilon}{2}L_s^{\bar{X}}(\varepsilon\mathbb{Z})\Big) \overset{d}{=} \sum_{k\in\mathbb{Z}}\tau^{\bar{Y}_k}\Big(\frac{\varepsilon}{2}L_s^{\bar{X}}(\varepsilon k)\Big), \tag{5.4}$$

where $\{\bar{Y}_k\}_{k\in\mathbb{Z}}$ are a family of independent reflected Brownian motions on $[0,1]$. That is, the time change $\psi^{\bar{X},\varepsilon}(t)$ has the same law as

$$\tilde{\psi}^{\bar{X},\varepsilon}(t) = \inf\big\{s>0 \mid s + \sum_{k\in\mathbb{Z}}\tau^{\bar{Y}_k}\Big(\frac{\varepsilon}{2}L_s^{\bar{X}}(\varepsilon k)\Big) > t\big\}\,. \tag{5.5}$$

Each of the processes $\tau^{\bar{Y}_k}$ can be represented as in (5.3) with independent Poisson random measures $N^{\bar{Y}_k}$:

$$\tau_\ell^{\bar{Y}_k} = \int_{[0,\ell]}\int_{[0,\infty)} sN^{\bar{Y}_k}(d\ell\times ds). \tag{5.6}$$

Since each of the random measures $N^{\bar{Y}_k}$ is atomic, we may define $\{(\ell_{j,k}, s_{j,k})\}_{j=1}^\infty$ to be the random atoms of $N^{\bar{Y}_k}$ by

$$N^{\bar{Y}_k} = \sum_{j=1}^\infty \delta_{(\ell_{j,k},s_{j,k})}. \tag{5.7}$$

Then define a random measure on $\mathbb{R}\times[0,\infty)$:

$$\mu^{\bar{X},\varepsilon} = dx\times d\ell + \sum_{k\in\mathbb{Z}}\sum_{j=1}^\infty s_{j,k}\delta_{(\varepsilon k,(2/\varepsilon)\ell_{j,k})} \tag{5.8}$$

Returning (5.5), we now have the representation

$$s + \sum_{k\in\mathbb{Z}}\tau^{\bar{Y}_k}\left(\frac{\varepsilon}{2}L_s^{\bar{X}}(\varepsilon k)\right) = \mu^{\bar{X},\varepsilon}\left(\{(x,\ell)\in\mathbb{R}\times[0,\infty) \mid \ell\leqslant L_s^{\bar{X}}(x)\}\right).$$

It is easy to check that $\mu^{\bar{X}}$ defines a Lévy trap measure, in the sense of [BAC+15], Definition 4.10. This proves the following:

Proposition 5.2. *Let $\bar{X}$ be a standard Brownian motion on $\mathbb{R}$ and let $\bar{X}[\mu^{\bar{X},\varepsilon}]$ be the trapped Brownian motion (see §2.4) with trap measure $\mu^{\bar{X},\varepsilon}$ defined by (5.8). Then the law of X^ε coincides with the law of $\bar{X}[\mu^{\bar{X},\varepsilon}]$.*

The process Y^ε admits a similar representation as a trapped (reflected) Brownian motion. To this end, we first note that $\tau_\ell^{\bar{X},\varepsilon}$ is also a Lévy subordinator which may be written as

$$\tau_\ell^{\bar{X},\varepsilon} = \int_{[0,\ell]}\int_{[0,\infty)} sN^{\bar{X},\varepsilon}(d\ell\times ds), \tag{5.9}$$

where $N^{\bar{X},\varepsilon}$ is a Poisson random measure on $[0,\infty)\times[0,\infty)$ with intensity measure $d\ell\times\eta^{\bar{X},\varepsilon}(s)ds$.

Lemma 5.3. *The excursion length measure $\eta^{\bar{X},\varepsilon}$ satisfies the scaling relation,*

$$\eta^{\bar{X},\varepsilon}(s) = \varepsilon^{-3}\eta^{\bar{X},1}(\varepsilon^{-2}s), \quad s>0.$$

Proof. First we need to compare $L_t^{\bar X}(\varepsilon\mathbb{Z})$ and $L_t^{\bar X}(\mathbb{Z})$. Here we use the Brownian scaling relation

$$\frac{1}{\varepsilon}\bar X_{\varepsilon^2 s} \stackrel{d}{=} \bar X_s$$

in distribution. We compute

$$\begin{aligned} L_t^{\bar X}(\varepsilon\mathbb{Z}) &= \lim_{\delta\to 0}\frac{1}{2\delta}\int_0^t \mathbf{1}_{\{d(\bar X_s,\varepsilon\mathbb{Z})<\delta\}}\,ds \\ &= \lim_{\delta\to 0}\frac{\varepsilon^2}{2\delta}\int_0^{t/\varepsilon^2} \mathbf{1}_{\{d(\frac{1}{\varepsilon}\bar X_{\varepsilon^2 s},\mathbb{Z})<\frac{\delta}{\varepsilon}\}}\,ds \\ &\stackrel{d}{=} \lim_{\delta\to 0}\frac{\varepsilon}{2\delta}\int_0^{t/\varepsilon^2} \mathbf{1}_{\{d(\bar X_s,\mathbb{Z})<\delta\}}\,ds \end{aligned}$$

where the last equality holds in distribution. Hence we obtain

$$L_t^{\bar X}(\varepsilon\mathbb{Z}) \stackrel{d}{=} \varepsilon L_{t/\varepsilon^2}^{\bar X}(\mathbb{Z}) \tag{5.10}$$

in distribution. Recall that $N^{\bar X,\varepsilon} = \sum_j \delta_{(s_j,\ell_j)}$ is a Poisson random measure with mass at (s_j,ℓ_j) if

$$\tau_{\ell_j}^{\bar X,\varepsilon} - \tau_{\ell_j -}^{\bar X,\varepsilon} = s_j$$

i.e. if $L_s^{\bar X}(\varepsilon\mathbb{Z}) \equiv \ell_j$ for an interval $s\in[t_0,t_0+s_j]$ where $t_0 = \tau_{\ell_j -}^{\bar X,\varepsilon}$. Therefore if we let

$$N^{\bar X,1} = \sum_j \delta_{(s_j,\ell_j)}$$

then (5.10) tells us

$$N^{\bar X,\varepsilon} = \sum_j \delta_{(\varepsilon^2 s_j,\varepsilon\ell_j)}\,.$$

So now let $[\bar s_0,\bar s_1]\times[\bar\ell_0,\bar\ell_1]\subset[0,\infty)\times[0,\infty)$. It follows that

$$\begin{aligned} \boldsymbol{E}\Big(N^{\bar X,\varepsilon}([\bar s_0,\bar s_1]\times[\bar\ell_0,\bar\ell_1])\Big) &= \boldsymbol{E}\Big(\sum_j \mathbf{1}_{\{(\varepsilon^2 s_j,\varepsilon\ell_j)\in[\bar s_0,\bar s_1]\times[\bar\ell_0,\bar\ell_1]\}}\Big) \\ &= \boldsymbol{E}\Big(N^{\bar X,1}([\bar s_0/\varepsilon^2,\bar s_1/\varepsilon^2]\times[\bar\ell_0/\varepsilon,\bar\ell_1/\varepsilon])\Big) \end{aligned}$$

Therefore since

$$\boldsymbol{E}\Big(N^{\bar X,\varepsilon}([\bar s_0,\bar s_1]\times[\bar\ell_0,\bar\ell_1])\Big) = \int_{\bar\ell_0}^{\bar\ell_1}\int_{\bar s_0}^{\bar s_1}\eta^{\bar X,\varepsilon}(s)\,d\ell ds\,,$$

we have

$$\int_{\bar\ell_0}^{\bar\ell_1}\int_{\bar s_0}^{\bar s_1}\eta^{\bar X,\varepsilon}(s)\,d\ell ds = \int_{\bar\ell_0/\varepsilon}^{\bar\ell_1/\varepsilon}\int_{\bar s_0/\varepsilon^2}^{\bar s_1/\varepsilon^2}\eta^{\bar X,1}(s)\,d\ell ds = \int_{\bar\ell_0}^{\bar\ell_1}\int_{\bar s_0}^{\bar s_1}\varepsilon^{-3}\eta^{\bar X,\varepsilon}\Big(\frac{s}{\varepsilon^2}\Big)\,d\ell ds$$

as desired. □

Letting $\{(s_j,\ell_j)\}_{j=1}^\infty$ denote the atoms of $N^{\bar X,\varepsilon}$ we then define a random measure on $[0,1]\times[0,\infty)$ by

$$\mu^{\bar Y,\varepsilon} = dy\times d\ell + \sum_{j=1}^\infty s_j\delta_{(0,(\varepsilon/2)\ell_j)}. \tag{5.11}$$

This also is a Lévy Trap Measure in the sense of [BAC+15] (replacing $\mathbb{R}$ by $[0,1]$).

Proposition 5.4. *Let $\bar{Y}$ be a Brownian motion on $[0,1]$, reflected at the endpoints, and let $\bar{Y}[\mu^{\bar{Y},\varepsilon}]$ be the trapped Brownian motion with trap measure $\mu^{\bar{Y},\varepsilon}$ defined by (5.11). Then the law of Y^ε coincides with the law of $\bar{Y}[\mu^{\bar{Y},\varepsilon}]$.*

5.3. **Convergence as $\varepsilon \to 0$.** Having identified X^ε and Y^ε as trapped Brownian motions, we can now describe their limit behavior with the help of Theorem 6.2 of [BAC+15].

Proposition 5.5. *Let $R(t)$ be a Brownian motion on $[0,1]$ reflected at both endpoints $x = 0, 1$, and B be a standard Brownian motion on $\mathbb{R}$.*

(1) As $\varepsilon \to 0$, we have $Y^\varepsilon \to Y$ vaguely in distribution on $D([0,\infty))$. Here $Y = R[\mu_*^{\bar{Y}}]$ is a reflected Brownian motion that is sticky at 0.

(2) As $\varepsilon \to 0$, we have $X^\varepsilon \to B[\mu_*^{\bar{X}}]$ vaguely in distribution on $D([0,\infty))$. The limit process here may also be written as $X(t) = B(2L_t^Y(0))$.

Remark 5.6. Using the SDE methods in §4 we are able to obtain joint convergence of the pair $(X^\varepsilon, Y^\varepsilon)$ (Theorem 4.2). The trapped Brownian motion framework here, however, only provides convergence of the processes X^ε and Y^ε individually.

Proposition 5.5 can be proved quickly from the following lemma.

Lemma 5.7. *Let $N_*^{\bar{Y}}$ be a Poisson random measure on $\mathbb{R} \times [0,\infty) \times [0,\infty)$ with intensity measure $dx \times d\ell \times \eta^{\bar{Y}}(s)\, ds$. As $\varepsilon \to 0$, the random measures $\mu^{\bar{X},\varepsilon}$ on $\mathbb{R} \times [0,\infty)$, defined by (5.8), converge vaguely in distribution to the random measure μ_*^X defined by*

$$\mu_*^X(A) = \int_{\mathbb{R}}\int_0^\infty \mathbf{1}_A(x,\ell)dx\, d\ell + \frac{1}{2}\int_{\mathbb{R}}\int_0^\infty \int_0^\infty \mathbf{1}_A(x,\ell) s N_*^{\bar{Y}}(dx \times d\ell \times ds)\ ,$$

for all $A \in \mathcal{B}(\mathbb{R} \times [0,\infty))$. The random measures $\mu^{\bar{Y},\varepsilon}$ on $[0,1] \times [0,\infty)$, defined by (5.11), converge vaguely in distribution to the measure μ_*^Y defined by

$$\mu_*^Y(A) = \int_0^1\int_0^\infty \mathbf{1}_A(y,\ell)dy\, d\ell + 2\int_0^\infty \mathbf{1}_A(0,\ell)\, d\ell \qquad A \in \mathcal{B}([0,1] \times [0,\infty))\,.$$

Proof. It suffices to show for rectangles $A = [x_0, x_1] \times [\ell_0, \ell_1]$ that

$$\mu^{\bar{X},\varepsilon}(A) \to \mu_*^X(A)$$

in distribution. We calculate the characteristic function using [Kyp06, Theorem 2.7],

$$\begin{aligned}
\boldsymbol{E}[e^{i\beta\mu^{\bar{X},\varepsilon}(A)}] &= \exp\Big(i\beta|A| + \sum_{\varepsilon k \in [x_0,x_1]} \int_{\frac{\varepsilon}{2}\ell_0}^{\frac{\varepsilon}{2}\ell_1} \int_0^\infty (1 - e^{i\beta s})\eta^{\bar{Y}}(s)\, ds\Big) \\
&= \exp\Big(i\beta|A| + \Big(\Big\lfloor \frac{x_1}{\varepsilon}\Big\rfloor - \Big\lceil \frac{x_0}{\varepsilon}\Big\rceil\Big)\frac{\varepsilon(\ell_1 - \ell_0)}{2}\int_0^\infty (1 - e^{i\beta s})\eta^{\bar{Y}}(s)\, ds\Big) \\
&\to \exp\Big(i\beta|A| + \frac{|A|}{2}\int_0^\infty (1 - e^{i\beta s})\eta^{\bar{Y}}(s)\, ds\Big)
\end{aligned}$$

as $\varepsilon \to 0$. We note that this last formula is the characteristic function for $\mu_\star^X(A)$. The calculation for $\mu^{\bar{Y},\varepsilon}(A)$ uses Lemma 5.3 and a change of variables as follows

$$\boldsymbol{E}[e^{i\beta\mu^{\bar{Y},\varepsilon}(A)}] = \exp\Big(i\beta|A| + \mathbf{1}_{[y_0,y_1]}(0)\int_{\frac{2}{\varepsilon}\ell_0}^{\frac{2}{\varepsilon}\ell_1}\int_0^\infty (1 - e^{i\beta s})\eta^{\bar{X},\varepsilon}(s)\, ds\Big)$$

$$= \exp\Big(i\beta|A| + \mathbf{1}_{[y_0,y_1]}(0)\frac{2(\ell_1-\ell_0)}{\varepsilon^4}\int_0^\infty (1-e^{i\varepsilon^2\beta s})\eta^{\bar{X},1}(\varepsilon^{-2}s)\,ds\Big)$$
$$= \exp\Big(i\beta|A| + \mathbf{1}_{[y_0,y_1]}(0)\frac{2(\ell_1-\ell_0)}{\varepsilon^2}\int_0^\infty (1-e^{i\varepsilon^2\beta s})\eta^{\bar{X},1}(s)\,ds\Big)\,.$$

Notice that by switching the integrals, we find

$$\frac{1}{\varepsilon^2}\int_0^\infty (1-e^{i\beta\varepsilon^2 s})\eta^{\bar{X},1}(s)\,ds = \frac{1}{\varepsilon^2}\int_0^\infty (-\beta i\varepsilon^2\int_0^s e^{i\beta\varepsilon^2 r}\,dr)\eta^{\bar{X},1}(s)\,ds$$
$$= \int_0^\infty e^{i\beta\varepsilon^2 r}\int_r^\infty \eta^{\bar{X},1}(s)\,ds\,dr\,.$$

Since $\eta^{\bar{X},1}$ has exponential tails, we can send $\varepsilon \to 0$, use dominated convergence and switch the integrals again to find

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_0^\infty (1-e^{i\beta\varepsilon^2 s})\eta^{\bar{X},1}(s)\,ds = \int_0^\infty s\eta^{\bar{X},1}(s)\,ds = 1$$

and hence

$$\boldsymbol{E}[e^{i\beta\mu^{\bar{Y},\varepsilon}(A)}] \to \boldsymbol{E}[e^{i\beta\mu_*^Y(A)}]\,.$$

□

Proof of Proposition 5.5. The convergence of Y^ε to $R[\mu_*^{\bar{Y}}]$ is an immediate consequence of Theorem 2.7, Lemma 5.7 above, and the properties of Poisson random measures. To identify the limiting process $R[\mu_*^{\bar{Y}}]$ as a sticky Brownian motion, observe that the time change has the form

$$\mu_*^{\bar{Y}}\left(\{(y,\ell)\in[0,1]\times[0,\infty)\;\big|\;L_s^R(y)\geqslant \ell\}\right) = s + 2L_s^R(0)\,.$$

Thus, the limit process is $Y(t) = R(\psi(t))$ where

$$\psi(t) = \inf\{s > 0 \mid s + 2L_s^R(0) > t\}\,.$$

This is precisely a sticky Brownian motion (see Lemma 3.1).

For the second assertion, the convergence of X^ε to $B[\mu_*^{\bar{X}}]$ is again an immediate consequence of Theorem 2.7 and Lemma 5.7 above. The fact that limiting process $B[\mu_*^{\bar{X}}]$ has the same law as X_t from Theorem 1.2 can be seen as follows. To compare the two, let us first write them in a similar form. If $L_t^{\bar{B}}(0)$ is the local time of $\bar{B}$ at 0, let $\tau_\ell^{\bar{B}}$ be the inverse

$$\tau_\ell^{\bar{B}} = \inf\{t > 0 \mid L_t^{\bar{B}}(0) > \ell\}.$$

Then, we have

$$X_t = \bar{W}_{2L^{\bar{B}}_{T(t)}} = \bar{W}(h^{-1}(t))$$

where

$$h(r) = \inf\{r > 0 \mid r + \tau^{\bar{B}}_{r/2} > t\}$$

The fact that $2L^{\bar{B}}_{T(t)} = h^{-1}(t)$ follows from the definition of $T(t)$, which implies $2L^{\bar{B}}_{T(t)} + T(t) = t$.

Therefore, the two processes are

$$B[\mu_*^{\bar{X}}] = B(\phi^{-1}(t)) \qquad X_t = \bar{B}(h^{-1}(t))$$

where ϕ is:

$$\phi(r) = \phi[\mu_*, B]_r = \mu_*\left(\{(x,\ell)\in\mathbb{R}\times[0,\infty) \mid L_r^B(x)\geqslant \ell\}\right)$$

If $A_r^B = \{(x,\ell) \in \mathbb{R} \times [0,\infty) \mid L_r^B(x) \geqslant \ell\}$, then by definition of the trap measure μ_*,

$$\phi(r) = r + \frac{1}{2} \int\int\int_{A_r^B \times [0,\infty)} s N_*^{\bar{Y}} (dx \times d\ell \times ds) \,. \tag{5.12}$$

The last integral has the same law as $\tau_{r/2}^{\bar{B}}$. Hence, h and ϕ have the same law. Notice that h is independent of $\bar{W}$, while ϕ depends on B, through the local time in A_r^B. □

5.4. **Proof of Proposition 5.1.** To abbreviate the notation, we will write $L_t^{\bar{X}}$ and $L_t^{\bar{Y}}$ for $L_t^{\bar{X}}(\varepsilon\mathbb{Z})$ and $L_t^{\bar{Y}}(0)$, respectively; notice that $L_t^{\bar{X}}$ depends on ε while $L_t^{\bar{Y}}$ does not. Let $X^\varepsilon(t) = \bar{X}(\psi^{\bar{X},\varepsilon}(t))$ and $Y^\varepsilon(t) = \bar{Y}(\psi^{\bar{Y},\varepsilon}(t))$. The proof of Proposition 5.1 follows quickly from Itô's formula, and the following two lemmas:

Lemma 5.8. *For every $t \geqslant 0$, we have*

$$L_t^{X^\varepsilon} = \frac{2}{\varepsilon} L_t^{Y^\varepsilon} \,. \tag{5.13}$$

Lemma 5.9. *The joint quadratic variation of X^ε and Y^ε is 0.*

Momentarily postponing the proof of these lemmas, we prove Proposition 5.1.

Proof of Proposition 5.1. For any $f \in \mathcal{D}(\mathcal{L}^\varepsilon)$, Itô's formula gives

$$\begin{aligned}
\boldsymbol{E} f(Z_t^\varepsilon) - f(Z_0^\varepsilon) &= \frac{1}{2} \boldsymbol{E} \int_0^{\psi^{\bar{X},\varepsilon}(t)} \partial_x^2 f(\bar{X}_s, \bar{Y}_s) \mathbf{1}_{\bar{X}_s \notin \varepsilon\mathbb{Z}} \, ds \\
&\quad + \frac{1}{2} \boldsymbol{E} \int_0^t \Big(\partial_x f((X_s^\varepsilon)^+, Y_s^\varepsilon) - \partial_x f((X_s^\varepsilon)^-, Y_s^\varepsilon) \Big) \, dL_s^{X^\varepsilon}(\varepsilon\mathbb{Z}) \\
&\quad + \frac{1}{2} \boldsymbol{E} \int_0^{\psi^{\bar{Y},\varepsilon}(t)} \partial_y^2 f(\bar{X}_s, \bar{Y}_s) \mathbf{1}_{\bar{Y}_s \in (0,1)} \, ds \\
&\quad + \boldsymbol{E} \int_0^t \partial_y f(X_s^\varepsilon, (Y_s^\varepsilon)^+) \, dL_s^{Y^\varepsilon}(0) \,.
\end{aligned}$$

Here we used the fact that $\langle X^\varepsilon, Y^\varepsilon \rangle = 0$ (Lemma 5.9) and $\partial_y f(x,1) = 0$ (which is guaranteed by the assumption $f \in \mathcal{D}(\mathcal{L}^\varepsilon)$). Using (5.13) this simplifies to

$$\begin{aligned}
\boldsymbol{E} f(Z_t^\varepsilon) - f(Z_0^\varepsilon) &= \boldsymbol{E} \int_0^{\psi^{\bar{X},\varepsilon}(t)} \partial_x^2 f(\bar{X}_s, \bar{Y}_s) \mathbf{1}_{\{\bar{X}_s \notin \varepsilon\mathbb{Z}\}} \, ds \\
&\quad + \boldsymbol{E} \int_0^{\psi^{\bar{Y},\varepsilon}(t)} \partial_y^2 f(\bar{X}_s, \bar{Y}_s) \mathbf{1}_{\{\bar{Y}_s \in (0,1)\}} \, ds \\
&\quad + \frac{1}{2} \boldsymbol{E} \int_0^t \Big(\partial_x^+ f(X_s^\varepsilon, Y_s^\varepsilon) - \partial_x^- f(X_s^\varepsilon, Y_s^\varepsilon) + \varepsilon \partial_y^+ f(X_s^\varepsilon, Y_s^\varepsilon) \Big) \, dL_s^{X^\varepsilon}(\varepsilon\mathbb{Z}) \,.
\end{aligned}$$

Since $f \in \mathcal{D}(\mathcal{L}^\varepsilon)$ and L^{X^ε} only increases when $Y^\varepsilon = 0$ and $X^\varepsilon \in \varepsilon\mathbb{Z}$, the last integral above vanishes. Consequently,

$$\lim_{t \to 0} \frac{1}{t} \boldsymbol{E}^{x,y} \big(f(Z_t^\varepsilon) - f(Z_0^\varepsilon) \big) = \mathcal{L}^\varepsilon f(x,y)$$

finishing the proof. □

It remains to prove Lemmas 5.8 and 5.9.

Proof of Lemma 5.8. We first claim that for any $t \geqslant 0$, we have

$$\psi^{\bar{X},\varepsilon}(t) + \psi^{\bar{Y},\varepsilon}(t) = t\,. \tag{5.14}$$

To see this, define the non-decreasing, right continuous function

$$H(t) \stackrel{\text{def}}{=} \tau^{\bar{Y}}\left(\frac{\varepsilon}{2} L_t^{\bar{X}}(\varepsilon\mathbb{Z})\right).$$

Using the properties of $\tau^{\bar{Y}}$, $L^{\bar{X}}$, $\tau^{\bar{X},\varepsilon}$, and $L^{\bar{Y}}$, it is easy to check that the right continuous inverse of H is

$$H^{-1}(t) = \inf\{s > 0 \mid \ H(s) > t\} = \tau^{\bar{X},\varepsilon}\left(\frac{2}{\varepsilon} L_s^{\bar{Y}}(0)\right).$$

Therefore, $\psi^{\bar{X},\varepsilon}$ and $\psi^{\bar{Y},\varepsilon}$ are the right continuous inverse functions of $t \mapsto t + H(t)$ and $t \mapsto t + H^{-1}(t)$, respectively, meaning that

$$\psi^{\bar{X},\varepsilon}(t) = \inf\{s \mid \ s + H(s) > t\}\,,$$
$$\psi^{\bar{Y},\varepsilon}(t) = \inf\left\{r \mid \ r + H^{-1}(r) > t\right\}.$$

In general, $H(H^{-1}(r)) \geqslant r$ and $H^{-1}(H(s)) \geqslant s$ must hold, but equality may not hold due to possible discontinuities in H and H^{-1}.

Fix $t > 0$, and let $[t_0, t_1]$ be the maximal interval such that $t \in [t_0, t_1]$ and $\psi^{\bar{X},\varepsilon}$ is constant on the interval $[t_0, t_1]$. Possibly $t_0 = t_1 = t$, but let us first suppose that the interval has non-empty interior, $t_0 < t_1$. This implies that $H(s)$ has a jump discontinuity at a point $s = \psi^{\bar{X},\varepsilon}(t_1)$ such that $s + H(s^-) = t_0$ and $s + H(s^+) = s + H(s) = t_1$. Also, $H^{-1}(H(s)) = s$ must hold for such a value of s. So, for $\ell = H(s) = H(\psi^{\bar{X},\varepsilon}(t_1))$ we have

$$\ell + H^{-1}(\ell) = H(s) + s = t_1.$$

Therefore, $\psi^{\bar{Y},\varepsilon}(t_1) = \ell$, since

$$\psi^{\bar{Y},\varepsilon}(t_1) = \inf\left\{r \mid \ r + H^{-1}(r) > t_1\right\}.$$

This means that $\psi^{\bar{Y},\varepsilon}(t_1) = H(s)$. Therefore,

$$\psi^{\bar{Y},\varepsilon}(t_1) + \psi^{\bar{X},\varepsilon}(t_1) = H(s) + s = t_1$$

must hold. Now let extend the equality to the rest of the interval $[t_0, t_1]$. By assumption, $\psi^{\bar{X},\varepsilon}(t) = \psi^{\bar{X},\varepsilon}(t_1)$ for all $t \in [t_0, t_1]$. Since H has a jump discontinuity at s, this means $H^{-1}(r)$ is constant on the interval $[H(s^-), H(s)]$. Hence, the function $r + H^{-1}(r)$ is affine with slope 1 on the interval $[H(s^-), H(s)] = [\psi^{\bar{Y},\varepsilon}(t_1) - (t_1 - t_0), \psi^{\bar{Y},\varepsilon}(t_1)]$. Therefore, for all $t \in [t_0, t_1]$, we must have

$$\psi^{\bar{Y},\varepsilon}(t) = \psi^{\bar{Y},\varepsilon}(t_1) + t - t_1.$$

This shows that for all $t \in [t_0, t_1]$, we have

$$\psi^{\bar{X},\varepsilon}(t) + \psi^{\bar{Y},\varepsilon}(t) = \psi^{\bar{X},\varepsilon}(t_1) + \psi^{\bar{Y},\varepsilon}(t_1) + t - t_1 = t \tag{5.15}$$

Applying the same argument with the roles of $\psi^{\bar{X},\varepsilon}$, $\psi^{\bar{Y},\varepsilon}$, H and H^{-1} reversed, we conclude that $\psi^{\bar{X},\varepsilon}(t) + \psi^{\bar{Y},\varepsilon}(t) = t$ must hold if either $\psi^{\bar{X},\varepsilon}$ or $\psi^{\bar{Y},\varepsilon}$ is constant on an interval containing t which has non-empty interior. The only other possibility is that both $\psi^{\bar{X},\varepsilon}$ and $\psi^{\bar{Y},\varepsilon}$ are strictly increasing through t. In this case, H must be continuous at $\psi^{\bar{X},\varepsilon}(t)$ and H^{-1} must be continuous at $\psi^{\bar{Y},\varepsilon}(t)$. Thus,

$H^{-1}(H(\psi^{\bar{X},\varepsilon}(t))) = \psi^{\bar{X},\varepsilon}(t)$ and $H(H^{-1}(\psi^{\bar{Y},\varepsilon}(t))) = \psi^{\bar{Y},\varepsilon}(t)$ holds. The rest of the argument goes as in the previous case. This proves (5.14).

Now, since X^ε and Y^ε are time changes of $\bar{X}$ and $\bar{Y}$ respectively, we know that the local times are given by

$$L_t^{X^\varepsilon} \stackrel{\text{def}}{=} L^{X^\varepsilon}(\varepsilon\mathbb{Z}) = L^{\bar{X}}_{\psi^{\bar{X},\varepsilon}(t)}, \quad \text{and} \quad L_t^{Y^\varepsilon} \stackrel{\text{def}}{=} L^{Y^\varepsilon}(0) = L^{\bar{Y}}_{\psi^{\bar{Y},\varepsilon}(t)}.$$

By definition of $\psi^{\bar{X},\varepsilon}$, we know

$$t = \psi^{\bar{X},\varepsilon}(t) + \tau^{\bar{Y}}\Big(\frac{\varepsilon}{2}L^{\bar{X}}(\psi^{\bar{X},\varepsilon}(t))\Big).$$

Using (5.14) this gives

$$\psi^{\bar{Y},\varepsilon}(t) = \tau^{\bar{Y}}\Big(\frac{\varepsilon}{2}L^{\bar{X}}(\psi^{\bar{X},\varepsilon}(t))\Big),$$

and using the fact that $\tau^{\bar{Y}}$ is the inverse of $L^{\bar{Y}}$, we get (5.13) as desired. □

Proof of Lemma 5.9. Without loss of generality, suppose that $(X_0^\varepsilon, Y_0^\varepsilon) = (0,0)$. Fix $\delta > 0$, and define a sequence of stopping times $0 = \sigma_0 < \theta_1 < \sigma_1 < \theta_2 < \sigma_2 < \dots$ inductively, by

$$\begin{aligned}
\sigma_0 &= 0\\
\theta_{k+1} &= \inf\{t > \sigma_k \mid \text{either } Y_t^\varepsilon = \delta \text{ or } d(X_t^\varepsilon, \varepsilon\mathbb{Z}) = \delta\}, \quad k = 0,1,2,3,\dots\\
\sigma_{k+1} &= \inf\{t > \theta_k \mid Y_t = 0 \text{ and } X_t^\varepsilon \in \varepsilon\mathbb{Z}\}, \quad k = 0,1,2,3,\dots
\end{aligned}$$

Then for $T > 0$, we decompose the joint quadratic variation over $[0,T]$ as

$$\langle X^\varepsilon, Y^\varepsilon\rangle_{[0,T]} = \sum_{k\geqslant 0} \langle X^\varepsilon, Y^\varepsilon\rangle_{[\sigma_k\wedge T, \theta_{k+1}\wedge T]} + \langle X^\varepsilon, Y^\varepsilon\rangle_{[\theta_{k+1}\wedge T, \sigma_{k+1}\wedge T]}$$

We claim that for all k,

$$\langle X^\varepsilon, Y^\varepsilon\rangle_{[\theta_{k+1}\wedge T, \sigma_{k+1}\wedge T]} = 0 \tag{5.16}$$

holds with probability one. Hence,

$$\begin{aligned}
\big|\langle X^\varepsilon, Y^\varepsilon\rangle_{[0,T]}\big| &\leqslant \sum_{k\geqslant 0} \big|\langle X^\varepsilon, Y^\varepsilon\rangle_{[\sigma_k\wedge T, \theta_{k+1}\wedge T]}\big|\\
&\leqslant \sum_{k\geqslant 0} \frac{1}{2}\langle X^\varepsilon, X^\varepsilon\rangle_{[\sigma_k\wedge T, \theta_{k+1}\wedge T]} + \frac{1}{2}\langle Y^\varepsilon, Y^\varepsilon\rangle_{[\sigma_k\wedge T, \theta_{k+1}\wedge T]}\\
&\leqslant \sum_{k\geqslant 0} |(\theta_{k+1}\wedge T) - (\sigma_k \wedge T)|\\
&\leqslant \Big|\Big\{t \in [0,T] \mid \quad |\bar{Y}_t| \leqslant \delta, \ \text{ and } \ d(\bar{X}_t, \varepsilon\mathbb{Z}) \leqslant \delta \quad \Big\}\Big| \qquad (5.17)
\end{aligned}$$

As $\delta \to 0$, the latter converges to 0 almost surely, which proves that $\langle X^\varepsilon, Y^\varepsilon\rangle = 0$.

To establish the claim (5.16), we may assume $\theta_k < T$, for otherwise, the statement is trivial. At time θ_k, we have either $X_{\theta_k}^\varepsilon \notin \varepsilon\mathbb{Z}$ or $Y_{\theta_k} = \delta$. In the former case, we must have $X_t \notin \varepsilon\mathbb{Z}$ for all $t \in [\theta_k, \sigma_k)$. Hence, $\psi^{\bar{Y},\varepsilon}(t)$ and Y_t^ε are constant for all $t \in [\theta_k, \sigma_k)$. In the other case, $Y_t > 0$ for all $t \in [\theta_k, \sigma_k)$ while X_t is constant on $[\theta_k, \sigma_k]$. In either case, this implies that $\langle X^\varepsilon, Y^\varepsilon\rangle_{[\theta_k\wedge T, \sigma_k\wedge T]} = 0$ holds with probability one. □

6. **The Fat Comb Model**

Recall the fat comb model defined in §1.4, which is the normally reflected diffusion Z^ε on the unbounded domain

$$\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid -\varepsilon < y < \mathbf{1}_{B(\varepsilon\mathbb{Z}, \varepsilon^2/2)}(x)\}, \tag{6.1}$$

(see Theorem 2.8). The same result as Theorem 4.2 holds for this process:

Theorem 6.1. *Let $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ be a normally reflected Brownian motion in Ω_ε with initial distribution μ^ε. If the sequence of measures (μ^ε) converges weakly to a probability measure μ on $\Omega_0 = \mathbb{R} \times [0, 1]$, then the processes Z^ε converge weakly (as $\varepsilon \to 0$) to the process $Z \stackrel{\text{def}}{=} (X, Y)$ which is a weak solution to the system (3.2)-(3.4) with initial distribution μ.*

To prove Theorem 6.1, we will need to establish the analogues of Lemmas 4.3 and 4.4. These are as follows:

Lemma 6.2. *Let $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ be the reflected Brownian motion on the fat comb Ω_ε, as described in Theorem 6.1. Then, for any $T > 0$, the family of processes Z^ε are tight in $C([0, T]; \mathbb{R}^2)$.*

Lemma 6.3. *If $f \in \mathcal{D}(A)$, and $K \subset \Omega_0$ is compact, then*

$$\lim_{\varepsilon\to 0} \sup_{z\in K\cap\Omega_\varepsilon} \boldsymbol{E}^z \Big(f((Z_t^\varepsilon)^+) - f((Z_0^\varepsilon)^+) - \int_0^t Af((Z_s^\varepsilon)^+)\, ds \Big) = 0$$

Proof of Theorem 6.1. Given Lemmas 6.2 and 6.3 above, the proof of Theorem 6.1 is identical to that of Theorem 4.2. □

Lemmas 6.2 and 6.3 are proved below in Sections 6.1 and 6.2 respectively.

6.1. **Proof of Tightness (Lemma 6.2).**

To prove tightness, we first compare the oscillation of trajectories in the spine to that of Brownian motion. This will also be used in the proof of Lemma 6.3.

Lemma 6.4. *Let W' be a standard Brownian motion on $\mathbb{R}$ with $W'(0) = 0$. For any $T > 0$, $\varepsilon \in (0, 1/2]$, $z \in \Omega_\varepsilon$, and any $a, \delta > 0$, we have*

$$\boldsymbol{P}^z \Big(\sup_{\substack{r,t\in[0,T] \\ |t-r|\leqslant\delta}} |X^\varepsilon(t) - X^\varepsilon(r)| \geqslant a \Big) \leqslant \boldsymbol{P} \Big(\sup_{\substack{r,t\in[0,T] \\ |t-r|\leqslant\delta}} 4|W'(t) - W'(r)| \geqslant a - 2\varepsilon \Big). \tag{6.2}$$

Proof. Define a sequence of stopping times inductively, by

$$\tau_0 = \inf\big\{t \geqslant 0 \bigm| X^\varepsilon(t) \in \varepsilon\big(\mathbb{Z} + \frac{1}{2}\big)\big\}$$
$$\tau_{k+1} = \inf\big\{t \geqslant \tau_k \bigm| |X^\varepsilon(t) - X^\varepsilon(\tau_k)| = \varepsilon\big\}, \quad k \geqslant 0.$$

By symmetry of the domain, observe that $k \mapsto X^\varepsilon(\tau_k)$ defines a simple random walk on the discrete points $\varepsilon(\mathbb{Z} + 1/2)$. Next, define

$$\tau_k' = \inf\{t \geqslant \tau_k \mid |X^\varepsilon(t) - X^\varepsilon(\tau_k)| = \varepsilon/4\}, \quad k \geqslant 0.$$

In particular, $\tau_k < \tau_k' < \tau_{k+1}$. At time τ_k, $X^\varepsilon(\tau_k)$ is in the spine, at the midpoint between two adjacent teeth. For $t \in [\tau_k, \tau_k']$, $X^\varepsilon(t)$ is in the spine and cannot enter the teeth, because $|X^\varepsilon(t) - x| \leqslant \varepsilon/4$ where $x = X^\varepsilon(\tau_k) \in \varepsilon(\mathbb{Z} + \frac{1}{2})$. Define the increments $\Delta_k X^\varepsilon = X^\varepsilon(\tau_{k+1}) - X^\varepsilon(\tau_k) \in \{-\varepsilon, +\varepsilon\}$. By the strong Markov property and symmetry of the domain, the random variables $\{(\tau_k' - \tau_k)\}_k \cup \{\Delta X_k^\varepsilon\}_k$ are independent.

Now, suppose that $W'(t)$ is an independent Brownian motion on $\mathbb{R}$, with $W'(0) = 0$. Define another set of stopping times inductively by $\sigma_0 = 0$ and

$$\sigma_{k+1} = \inf\{t \geqslant \sigma_k \mid |W'(t) - W'(\sigma_k)| = \varepsilon/4\}, \quad k \geqslant 0. \tag{6.3}$$

Let $\Delta\sigma_k = \sigma_{k+1} - \sigma_k$, and $\Delta_k W' = W'(\sigma_{k+1}) - W'(\sigma_k) \in \{-\varepsilon/4, \varepsilon/4\}$. Observe that the family of random variables

$$\{(\sigma_{k+1} - \sigma_k), 4\Delta W_k'\}_{k\geqslant 0}$$

has the same law as the family

$$\{(\tau_k' - \tau_k), \Delta X_k^\varepsilon\}_{k\geqslant 0}.$$

Next, define

$$K(t) = \max\{k \geqslant 0 \mid \tau_k \leqslant t\},$$

and observe that if $|t-r| \leqslant \delta$ and $0 \leqslant r \leqslant t \leqslant T$, then we must have $\tau_{K(t)} - \tau_{K(r)+1} \leqslant \delta$ and thus

$$\sum_{j=K(r)+1}^{K(t)-1} (\tau_j' - \tau_j) \leqslant \delta, \qquad \text{and} \qquad \sum_{j=0}^{K(t)-1} (\tau_j' - \tau_j) \leqslant T.$$

In this case,

$$\begin{aligned} |X^\varepsilon(t) - X^\varepsilon(r)| &\leqslant 2\varepsilon + |X^\varepsilon(K(t)) - X^\varepsilon(K(r)+1)| \\ &= 2\varepsilon + \Big|\sum_{j=K(r)+1}^{K(t)-1} \Delta X_j^\varepsilon\Big| \\ &\leqslant 2\varepsilon + \sup_{0\leqslant \ell\leqslant m} \Big|\sum_{j=\ell+1}^{m-1} \Delta X_j^\varepsilon\Big| \mathbf{1}_{\left\{\sum_{j=\ell+1}^{m-1}(\tau_j'-\tau_j)\leqslant\delta\right\}} \mathbf{1}_{\left\{\sum_{j=0}^{m-1}(\tau_j'-\tau_j)\leqslant T\right\}}\,. \end{aligned} \tag{6.4}$$

This last supremum has the same law as

$$\begin{aligned} &\sup_{0\leqslant \ell\leqslant m} \Big|\sum_{j=\ell+1}^{m-1} 4\Delta W_j'\Big| \mathbf{1}_{\left\{\sum_{j=\ell+1}^{m-1}(\sigma_{j+1}-\sigma_j)\leqslant\delta\right\}} \mathbf{1}_{\left\{\sum_{j=0}^{m-1}(\sigma_{j+1}-\sigma_j)\leqslant T\right\}} \\ &\qquad = \sup_{0\leqslant \ell\leqslant m} 4|W'(\sigma_m) - W'(\sigma_{\ell+1})|\, \mathbf{1}_{\{\sigma_m - \sigma_{\ell+1}\leqslant\delta\}}\, \mathbf{1}_{\{\sigma_m-\sigma_0\leqslant T\}}\,. \end{aligned}$$

Since the right hand side of the above is bounded by

$$\sup_{\substack{r,t\in[0,T]\\|t-r|\leqslant\delta}} 4|W'(t) - W'(r)|\,,$$

we obtain (6.2). ◻

We now prove Lemma 6.2.

Proof of Lemma 6.2. Note first that Lemma 6.4 immediately implies that the processes X^ε are tight. Indeed, by (6.2) we see

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \boldsymbol{P}^{\mu^\varepsilon}\Big(\sup_{\substack{r,t\in[0,T]\\|t-r|\leqslant\delta}} |X^\varepsilon(t) - X^\varepsilon(r)| \geqslant a\Big) = 0\,. \tag{6.5}$$

Moreover, since μ^ε converge weakly to the probability measure μ, the distributions of X_0^ε are tight. This implies implies tightness of the processes X^ε.

For tightness of Y^ε, we note as above that the distributions of Y_0^ε are already tight. In order to control the time oscillations, fix $T > 0$, and let

$$dZ^\varepsilon = dB_t + dL_t^{\partial\Omega_\varepsilon} ,$$

be the semi-martingale decomposition of Z^ε (see for instance [SV71]). Here $B = (B_1, B_2)$ is a standard Brownian motion and $L^{\partial\Omega_\varepsilon}$ is the local time of Z^ε on $\partial\Omega_\varepsilon$. Let $\omega(\delta) = \omega_T(\delta)$, defined by

$$\omega(\delta) = \sup_{\substack{s,t\in[0,T] \\ |t-s|\leqslant\delta}} |B_2(t) - B_2(s)| ,$$

be the modulus of continuity for B_2 over $[0, T]$. Let $[s, t] \subset [0, T]$ with $|t - s| \leqslant \delta$. If $0 < Y_r^\varepsilon < 1$ for all $r \in (s, t)$, then we must have

$$|Y^\varepsilon(t) - Y^\varepsilon(s)| = |B_2(t) - B_2(s)| \leqslant \omega(\delta) .$$

Otherwise, for some $r \in (s, t)$ either $Y_r = 0$ or $Y_r = 1$. Let G_δ be the event that $\omega(\delta) < 1/2$; on this event Y cannot hit both 0 and 1 on the interval $[s, t]$. Define

$$\eta_- = \inf\{r > s \mid Y_r^\varepsilon \in \{0, 1\}\} , \quad \text{and} \quad \eta_+ = \sup\{r < t \mid Y_r^\varepsilon \in \{0, 1\}\} .$$

In this case we have

$$\begin{aligned} |Y_t^\varepsilon - Y_s^\varepsilon| &\leqslant \max(|Y^\varepsilon(\eta_-) - Y^\varepsilon(s)| \,,\ |Y^\varepsilon(t) - Y^\varepsilon(\eta_+)|) + \mathbf{1}_{G_\delta^c} + \varepsilon^2 \\ &= \max(|B(\eta_-) - B(s)| \,,\ |B(t) - B(\eta_+)|) + \mathbf{1}_{G_\delta^c} + \varepsilon^2 \leqslant \omega(\delta) + \mathbf{1}_{G_\delta^c} + \varepsilon^2 \end{aligned}$$

Combining the two cases, we see that for any $z \in \Omega_\varepsilon$,

$$\boldsymbol{P}^z\Big(\sup_{\substack{s,t\in[0,T] \\ |t-s|\leqslant\delta}} |Y^\varepsilon(t) - Y^\varepsilon(s)| > a\Big) \leqslant \boldsymbol{P}(\omega(\delta) > a - \varepsilon^2) + \boldsymbol{P}(G_\delta^c)$$

Since the right hand side is independent of z, integrating over z with respect to μ^ε implies

$$\lim_{\delta\to 0} \limsup_{\varepsilon\to 0} \boldsymbol{P}^{\mu^\varepsilon}\Big(\sup_{\substack{s,t\in[0,T] \\ |t-s|\leqslant\delta}} |Y^\varepsilon(t) - Y^\varepsilon(s)| > a\Big) = 0$$

holds for any $a > 0$. This shows tightness of Y^ε in $C([0, T])$, finishing the proof of Lemma 6.2. □

6.2. **Generator estimate.** The main idea behind proving Lemma 6.3 is to balance the local time Z^ε spends at the "gate" between the spine and teeth, and the time spent in the spine. Explicitly, let $S \stackrel{\text{def}}{=} \mathbb{R} \times (-\varepsilon, 0)$ denote the spine of Ω_ε, and T, defined by

$$T \stackrel{\text{def}}{=} \bigcup_{k\in\varepsilon\mathbb{Z}} \Big\{(x, y) \Bigm| |x - \varepsilon k| < \frac{\varepsilon^2}{2},\ y \in (0, 1)\Big\} ,$$

denote the collection of the teeth (see (1.14) and Figure 3). Let the "gate" G, defined by

$$G \stackrel{\text{def}}{=} \partial T \cap \partial S = \bigcup_{k\in\varepsilon\mathbb{Z}} \Big\{(x, 0) \Bigm| |x - \varepsilon k| \leqslant \frac{\varepsilon^2}{2}\Big\} ,$$

denote the union of short segments connecting the spine and teeth. Now the required local time balance can be stated as follows.

Lemma 6.5. *For every $g \in C_b^1(\mathbb{R})$ and $K \subseteq \Omega_0$ compact we have*

$$\lim_{\varepsilon \to 0} \sup_{z \in K \cap \Omega_\varepsilon} \boldsymbol{E}^z \Big(\int_0^t g(X_s^\varepsilon) \mathbf{1}_{\{Y_s^\varepsilon < 0\}} \, ds - 2 \int_0^t g(X_s^\varepsilon) dL_s^G \Big) = 0 \,. \tag{6.6}$$

Next, we will also need to show that the local times on the left edges and right edges of the teeth balance. Explicitly, let ∂T^- and ∂T^+ denote the left and right edges of the teeth, defined respectively by

$$\partial T^- \stackrel{\text{def}}{=} \Big\{ (x, y) \in Z^\varepsilon \;\Big|\; x \in \varepsilon\mathbb{Z} - \frac{\varepsilon^2}{2}, \; y > 0 \Big\} \,,$$

$$\text{and} \qquad \partial T^+ \stackrel{\text{def}}{=} \Big\{ (x, y) \in Z^\varepsilon \;\Big|\; x \in \varepsilon\mathbb{Z} + \frac{\varepsilon^2}{2}, \; y > 0 \Big\} \,.$$

Let L^+ and L^- be the local times of Z^ε about ∂T^- and ∂T^+ respectively, and let $L^\pm$ denote the difference

$$L^\pm = L^- - L^+ \,.$$

The balance on the teeth boundaries we require is as follows.

Lemma 6.6. *For every $f \in \mathcal{D}(A)$ and $K \subseteq \Omega_0$ compact, we have*

$$\lim_{\varepsilon \to 0} \sup_{z \in K \cap \Omega_\varepsilon} \boldsymbol{E}^z \Big(\int_0^t \frac{1}{2} \partial_x^2 f(Z_s^\varepsilon) \mathbf{1}_{\{Y_s^\varepsilon > 0\}} \, ds + \int_0^t \partial_x f(Z_s^\varepsilon) \, dL_s^\pm \Big) = 0 \,. \tag{6.7}$$

Momentarily postponing the proofs of Lemmas 6.5 and 6.6, we prove Lemma 6.3.

Proof of Lemma 6.3. Given $f \in \mathcal{D}(A)$, we define $f^\varepsilon \colon \Omega_\varepsilon \to \mathbb{R}$ by

$$f^\varepsilon(x, y) \stackrel{\text{def}}{=} f(x, y^+) \,.$$

Thus, $f((Z_t^\varepsilon)^+) = f^\varepsilon(Z_t^\varepsilon)$, and Lemma (6.2) reduces to showing

$$\lim_{\varepsilon \to 0} \sup_{z \in K \cap \Omega_\varepsilon} \boldsymbol{E}^z \Big(f^\varepsilon(Z_t^\varepsilon) - f^\varepsilon(Z_0^\varepsilon) - \int_0^t \frac{1}{2} \partial_y^2 f^\varepsilon(Z_s^\varepsilon) \, ds \Big) = 0 \,.$$

Since $f \in \mathcal{D}(A)$, we have $\partial_x^2 f(x, 0) + \partial_y f(x, 0) = \partial_y^2 f(x, 0)$ and $\partial_y f(x, 1) = 0$. Therefore, the extension f^ε satisfies $\partial_x^2 f^\varepsilon(x, y) = \partial_y^2 f^\varepsilon(x, 0^+) - \partial_y f^\varepsilon(x, 0^+)$ for $(x, y) \in S$, as well as $\partial_y f^\varepsilon = 0$ for $(x, y) \in S$. Notice that $\partial_y f^\varepsilon$ may be discontinuous across G. Using these facts and Itô's formula, we compute

$$\begin{aligned}
\boldsymbol{E}^z \Big(f^\varepsilon(Z_t^\varepsilon) - f^\varepsilon(Z_s^\varepsilon) \Big) &= \boldsymbol{E}^z \Big(\int_0^t \frac{1}{2} \left(\partial_y^2 f(Z_s^\varepsilon) + \partial_x^2 f(Z_s^\varepsilon) \right) \mathbf{1}_{\{Y_s^\varepsilon > 0\}} \, ds \Big) \\
&\quad + \boldsymbol{E}^z \Big(\int_0^t \frac{1}{2} \partial_x^2 f(X_s^\varepsilon, 0^+) \mathbf{1}_{\{Y_s^\varepsilon < 0\}} \, ds \Big) \\
&\quad + \boldsymbol{E}^z \Big(\int_0^t \partial_y f(X_s^\varepsilon, 0^+) \, dL_s^G + \int_0^t \partial_x f(Z_s^\varepsilon) \, dL_s^\pm \Big) \\
&= \boldsymbol{E}^z \Big(\int_0^t \frac{1}{2} \left(\partial_y^2 f(Z_s^\varepsilon) + \partial_x^2 f(Z_s^\varepsilon) \right) \mathbf{1}_{\{Y_s^\varepsilon > 0\}} \, ds \Big) \\
&\quad + \boldsymbol{E}^z \Big(\frac{1}{2} \int_0^t \left(\partial_y^2 f(X_s^\varepsilon, 0^+) - \partial_y f(X_s^\varepsilon, 0^+) \right) \mathbf{1}_{\{Y_s^\varepsilon < 0\}} \, ds \Big) \\
&\quad + \boldsymbol{E}^z \Big(\int_0^t \partial_y f(X_s^\varepsilon, 0^+) \, dL_s^G + \int_0^t \partial_x f(Z_s^\varepsilon) \, dL_s^\pm \Big) \,,
\end{aligned}$$

and hence

$$\boldsymbol{E}^z\Big(f^\varepsilon(Z^\varepsilon_t) - f^\varepsilon(Z^\varepsilon_0) - \int_0^t \frac{1}{2}\partial_y^2 f^\varepsilon(Z^\varepsilon_s)\,ds\Big)$$
$$= \boldsymbol{E}^z\Big(\int_0^t \frac{1}{2}\partial_x^2 f(Z^\varepsilon_s)\mathbf{1}_{\{Y^\varepsilon_s>0\}}\,ds + \int_0^t \partial_x^{\pm} f\,dL^T_s\Big)$$
$$- \frac{1}{2}\boldsymbol{E}^z\Big(\int_0^t \partial_y f(X^\varepsilon_s,0^+)\mathbf{1}_{\{Y^\varepsilon_s<0\}}\,ds - 2\int_0^t \partial_y f(X^\varepsilon_s,0^+)dL^G_s\Big)\,.$$

Using Lemmas 6.5 and 6.6 we see that the supremum over $z \in \Omega_\varepsilon \cap K$ of the right hand side of the above vanishes as $\varepsilon \to 0$. This proves Lemma 6.3. ☐

It remains to prove Lemmas 6.5 and 6.6, and we do this in subsequent sections.

6.3. **Local time at the gate (Lemmas 6.5).** The crux in the proof of Lemma 6.5 is an oscillation estimate on the solution to a specific Poisson equation with Neumann boundary conditions (Proposition 6.7, below). We state this when it is first encountered, and prove it in the next subsection.

Proof of Lemma 6.5. The expectation in (6.6) can be written as

$$\boldsymbol{E}^z\Big(\int_0^t g(X^\varepsilon_s)\mathbf{1}_{\{Y^\varepsilon_s<0\}}\,ds - 2\int_0^t g(X^\varepsilon_s)\,dL^G_s\Big) \tag{6.8}$$
$$= \sum_{k\in\mathbb{Z}} g(\varepsilon k)\boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{\{Y^\varepsilon_s<0\}}\mathbf{1}_{\{|X^\varepsilon_s-\varepsilon k|<\varepsilon/2\}}\,ds - 2\int_0^t \mathbf{1}_{\{|X^\varepsilon_s-\varepsilon k|<\varepsilon/2\}}\,dL^G_s\Big)$$
$$+ R^\varepsilon$$

where the remainder term R^ε is given by

$$R^\varepsilon \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} \boldsymbol{E}^z\Big(\int_0^t (g(X^\varepsilon_s) - g(\varepsilon k))\mathbf{1}_{\{Y^\varepsilon_s<0\}}\mathbf{1}_{\{|X^\varepsilon_s-\varepsilon k|<\varepsilon/2\}}\,ds\Big)$$
$$- 2\boldsymbol{E}^z\Big(\int_0^t (g(X^\varepsilon_s) - g(\varepsilon k))\mathbf{1}_{\{|X^\varepsilon_s-\varepsilon k|<\varepsilon/2\}}dL^G_s\Big) \stackrel{\text{def}}{=} R^\varepsilon_1 + R^\varepsilon_2\,.$$

To estimate R^ε, for any $\delta > 0$ we choose sufficiently large $M > 0$ such that

$$\sup_{(x,y)\in K} \boldsymbol{E}\Big(\int_0^t \mathbf{1}_{\{|x|+4|W_s|+2\geqslant M\}}\,ds\Big) < \frac{\delta}{\|g\|_\infty}\,, \tag{6.9}$$

where W is a standard Brownian motion in $\mathbb{R}$. By Lemma 6.4, we have

$$\boldsymbol{P}^z(|X^\varepsilon_s| + 1 \geqslant M) \leqslant \boldsymbol{P}(x + 4|W_s| + 2 \geqslant M)\,,$$

where $z = (x, y)$ and so the above estimate can be applied for X^ε independent of $\varepsilon \in (0, 1/2]$. Since g is continuous and hence uniformly continuous on $[-M, M]$, for any $\delta > 0$ we can choose $\varepsilon > 0$ such that if $x_1, x_2 \in [-M, M]$ with $|x_1 - x_2| < \varepsilon$ then $|g(x_1) - g(x_2)| < \delta$. For such ε and for integers $k \in \varepsilon^{-1}[-M, M]$ we have

$$\boldsymbol{E}^{(x,y)} \int_0^t |g(\varepsilon k) - g(X^\varepsilon_s)|\mathbf{1}_{\{Y^\varepsilon_s<0,\ |X^\varepsilon_s-\varepsilon k|<\varepsilon/2\}}\,ds \tag{6.10}$$
$$\leqslant \delta\int_0^t \boldsymbol{P}^z\Big(|X^\varepsilon_s - \varepsilon k| < \varepsilon/2\Big)\,ds\,.$$

Combining the above with (6.9), gives the following estimate of R_1^ε

$$|R_1^\varepsilon| \leqslant \Bigg(\delta \sum_{\substack{k\in\mathbb{Z}\\ \varepsilon k\in[-M,M]}} \int_0^t \boldsymbol{P}^z\Big(|X_s^\varepsilon - \varepsilon k| < \frac{\varepsilon}{2}\Big)\,ds$$

$$+ 2\|g\|_\infty \sum_{|\varepsilon k|>M} \int_0^t \boldsymbol{P}^z\Big(|X_s^\varepsilon - \varepsilon k| < \frac{\varepsilon}{2}\Big)\,ds\Bigg)$$

$$\leqslant (t+2)\delta\,.$$

Since $\delta > 0$ was arbitrary this proves $R_1^\varepsilon \to 0$ as $\varepsilon \to 0$. An estimate for R_2^ε can be obtained in the same manner. Namely,

$$|R_2^\varepsilon| \leqslant 2\bigg(\delta \boldsymbol{E}^z\big(L_t^G\big) + 2\|g\|_\infty \sum_{\substack{k\in\mathbb{Z}\\ |\varepsilon k|\geqslant M}} \boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{\{|X_s^\varepsilon - \varepsilon k|<\varepsilon/2\}}\,dL_s^G\Big)\bigg)$$

$$\leqslant c(t)\delta + 2\|g\|_\infty \boldsymbol{E}^x\Big(\int_0^t \mathbf{1}_{\{|X_s^\varepsilon|+1\geqslant M\}}\,dL_s^G\Big)\,.$$

Let $\tau = \inf\{t \mid |X_t^\varepsilon| + 1 \geqslant M\}$ and note that by the Markov property

$$\boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{\{|X_s^\varepsilon|+1\geqslant M\}}\,dL_s^G\Big) \leqslant \boldsymbol{E}^z\Big(\boldsymbol{E}^{X_\tau^\varepsilon}\Big(L_{t-t\wedge\tau}^G\Big)\Big)$$

$$\leqslant \Big(\sup_{z'} \boldsymbol{E}^{z'}\big(L_t^G\big)\Big)\boldsymbol{P}^z(\tau < t)\,.$$

Applying Itô's formula to the function

$$w(x,y) = \begin{cases} \frac{1}{2}(1-y)^2\,, & y\in[0,1]\,,\\ 0, & \text{otherwise,}\end{cases}$$

shows that

$$\boldsymbol{E}^z(L_t^G) = O(1)\,. \tag{6.11}$$

By choosing M larger, if necessary, we have

$$\sup_{z\in K} \boldsymbol{P}^z(\tau < t) < \delta$$

for all $\varepsilon \in (0, 1/2]$. Since $\delta > 0$ is arbitrary, this shows that $R_2^\varepsilon \to 0$ as $\varepsilon \to 0$.

Next, we need a PDE estimate to control the expression

$$\boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{\{Y_s^\varepsilon<0\}}\mathbf{1}_{\{|X_s-\varepsilon k|<\varepsilon/2\}}\,ds - 2\int_0^t \mathbf{1}_{\{|X_s-\varepsilon k|<\varepsilon/2\}} dL_s^G\Big).$$

from (6.8). To this end, let Q be a region of width ε directly below the tooth at $x = 0$, and G_0 be the which is the component of G contained in $[-\varepsilon/2, \varepsilon/2] \times \mathbb{R}$. Explicitly, let

$$Q \stackrel{\text{def}}{=} \Big[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\Big] \times \big[-\varepsilon, 0\big] \qquad \text{and} \qquad G_0 = \Big\{(x,0)\;\Big|\; -\frac{\varepsilon^2}{2} < x < \frac{\varepsilon^2}{2}\Big\}\,. \tag{6.12}$$

Let μ^ε denote the one dimensional Hausdorff measure of G_0.

Proposition 6.7. *Let the function* $u^\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ *be the solution of*

$$-\Delta u^\varepsilon = \mathbf{1}_Q - \mu^\varepsilon \qquad \text{in } \Omega_\varepsilon \tag{6.13}$$

$$\partial_\nu u^\varepsilon = 0 \qquad \text{on } \partial\Omega_\varepsilon\,, \tag{6.14}$$

with the normalization condition

$$\inf_{\Omega_\varepsilon} u^\varepsilon = 0\,. \tag{6.15}$$

Then there exists a constant $C > 0$*, independent of* ε *such that*

$$\sup_{\Omega_\varepsilon} u^\varepsilon(z) \leqslant C\varepsilon^2 |\ln \varepsilon|\,. \tag{6.16}$$

Throughout the remainder of this proof and the section, we will use the convention that $C > 0$ is a constant that is independent of ε. We apply Itô's formula to the function u^ε defined in Proposition 6.7 to obtain

$$2\boldsymbol{E}(u^\varepsilon(Z_t^\varepsilon) - u^\varepsilon(Z_0^\varepsilon)) = -\boldsymbol{E}\Big(\int_0^t \mathbf{1}_Q(Z_s^\varepsilon)\,ds - 2L_t^{G_0}\Big)\,.$$

$$= \boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{\{Y_s^\varepsilon<0\}}\mathbf{1}_{\{|X_s|<\varepsilon/2\}}\,ds - 2\int_0^t \mathbf{1}_{\{|X_s|<\varepsilon/2\}} dL_s^G\Big).$$

The oscillation bound (6.16) now implies

$$\Big|\boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{\{Y_s^\varepsilon<0\}}\mathbf{1}_{\{|X_s-\varepsilon k|<\varepsilon/2\}}\,ds - 2\int_0^t \mathbf{1}_{\{|X_s-\varepsilon k|<\varepsilon/2\}} dL_s^G\Big)\Big| \leqslant C\varepsilon^2 |\log \varepsilon|$$

holds for all k and $x \in \mathbb{R}$. Because of (6.9), we can restrict the sum in (6.8) to $k \in \mathbb{Z}$ for which $\varepsilon|k| \leqslant M$ (i.e. only $O(\varepsilon^{-1})$ terms in the sum). Therefore,

$$\sum_{\substack{k\in\mathbb{Z}\\ \varepsilon|k|\leqslant M}} \boldsymbol{E}\Big(\int_0^t \mathbf{1}_{\{Y_s^\varepsilon<0\}}\mathbf{1}_{\{|X_s^\varepsilon-\varepsilon k|<\varepsilon/2\}}\,ds - 2\int_0^t \mathbf{1}_{\{|X_s^\varepsilon-\varepsilon k|<\varepsilon/2\}} dL_s^G\Big)$$
$$\leqslant O(\varepsilon|\log(\varepsilon)|).$$

Combining this with the above estimates, we conclude that (6.6) holds. □

It remains to prove Proposition 6.7, which we do in the next subsection.

6.4. **An oscillation estimate for the Neumann problem (Proposition 6.7).** The proof of Proposition 6.7 involves a "geometric series" argument using the probabilistic representation. Explicitly, we obtain the desired oscillation estimate by estimating the probabilities of successive visits of Z^ε between two segments. The key step in the proof involves the so called narrow escape problem (see for instance [HS14]), which guarantees that the probability that Brownian motion exists from a given interval on the boundary of a domain vanishes logarithmically with the interval size. In our specific scenario, however, we can not directly use the results of [HS14] and we prove the required estimates here.

Proof of Proposition 6.7. Note first that

$$\int_{\Omega_\varepsilon} \big(\mathbf{1}_Q - \mu^\varepsilon\big)\,dz = 0\,,$$

and hence a bounded solution to (6.13)–(6.14) exists. Moreover, because the measure $\mathbf{1}_Q(z) - \mu^\varepsilon$ is supported in $\bar{Q}$, the function u^ε is harmonic in $\Omega_\varepsilon \setminus \bar{Q}$. Thus, by the maximum principle,

$$\sup_{\Omega_\varepsilon} u^\varepsilon \leqslant \sup_Q u^\varepsilon \,.$$

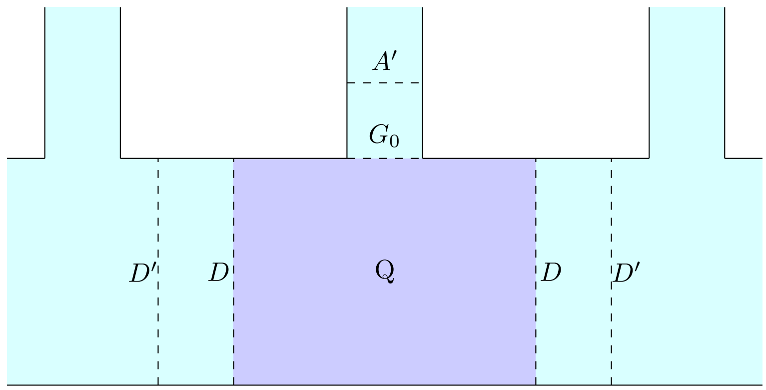

FIGURE 5. Image of one period of Ω_ε.

Define $Q' \supseteq Q$ to be the region that enlarges Q by ε^2 on the top, and $\varepsilon/4$ on the sides. Precisely, let

$$Q' \stackrel{\text{def}}{=} \Omega_\varepsilon \bigcap \Big(\big[-\frac{3\varepsilon}{4}, \frac{3\varepsilon}{4} \big] \times \big[-\varepsilon, \varepsilon^2 \big] \Big) \,.$$

The first step is to estimate the oscillation of u^ε on the top and side portion of Q'. Let A' and D', defined by

$$A' \stackrel{\text{def}}{=} \Big[-\frac{\varepsilon^2}{2}, \frac{\varepsilon^2}{2} \Big] \times \big\{ \varepsilon^2 \big\} \qquad \text{and} \qquad D' \stackrel{\text{def}}{=} \Big\{ \pm \frac{3\varepsilon}{4} \Big\} \times \big[-\varepsilon, 0 \big] \tag{6.17}$$

denotes the top and sides of Q' respectively. We aim to show

$$\sup_{a,d \in A' \cup D'} |u^\varepsilon(a) - u^\varepsilon(d)| \leqslant C\varepsilon^2 |\ln \varepsilon| \,. \tag{6.18}$$

Let τ_0 be the first time at which the process Z_t^ε hits the gate G_0 (defined in (6.12)). The stopping time τ_0 is finite almost surely, but has infinite expectation. We claim that the distribution of $Z_{\tau_0}^\varepsilon$ on G is bounded below by a constant multiple of the Hausdorff measure, uniformly over all initial points in $A' \cup D'$.

Lemma 6.8. *For any $z \in A' \cup D'$, let $\rho_0(z, \cdot)$, defined by*

$$\rho_0(z, r) = \boldsymbol{P}^z(Z_{\tau_0}^\varepsilon \in dr) \,,$$

denote the density of the random variable $Z_{\tau_0}^\varepsilon$ on G_0. Then, there exists $\delta > 0$ such that

$$\rho(z, r) \geqslant \frac{\delta}{\varepsilon^2} \,, \tag{6.19}$$

for all $z \in A' \cup D'$ and $r \in G_0$.

Momentarily postponing the proof of this lemma, we note that Lemma 6.8 implies that both

$$\rho(z,r) - \frac{\delta}{\varepsilon^2} \geqslant 0 \qquad \text{and} \qquad \int_{G_0} \Bigl(\rho(z,r) - \frac{\delta}{\varepsilon^2}\Bigr)\, dr = (1-\delta)\,. \tag{6.20}$$

Consequently, for any $a, d \in A' \cup D'$, we have

$$\begin{aligned}
\boldsymbol{E}^a u^\varepsilon(Z^\varepsilon_{\tau_0}) - \boldsymbol{E}^d u^\varepsilon(Z^\varepsilon_{\tau_0}) &= \int_{G_0} \rho(a,r) u^\varepsilon(r)\, dr - \int_{G_0} \rho(d,r) u^\varepsilon(r)\, dr \\
&= \int_{G_0} \Bigl(\rho(a,r) - \frac{\delta}{\varepsilon^2}\Bigr) u^\varepsilon(r)\, dr - \int_S \Bigl(\rho(d,r) - \frac{\delta}{\varepsilon^2}\Bigr) u^\varepsilon(r)\, dr \\
&\leqslant (1-\delta)\Bigl(\sup_{G_0} u^\varepsilon - \inf_{G_0} u^\varepsilon\Bigr) \leqslant (1-\delta)\Bigl(\sup_{r_1, r_2 \in G_0} |u^\varepsilon(r_1) - u^\varepsilon(r_2)|\Bigr)\,.
\end{aligned}$$

Now by Itô's formula,

$$\begin{aligned}
u^\varepsilon(a) - u^\varepsilon(d) &= \boldsymbol{E}^a u^\varepsilon(Z^\varepsilon_{\tau_0}) - \boldsymbol{E}^d u^\varepsilon(Z^\varepsilon_{\tau_0}) - \frac{1}{2}\boldsymbol{E}^a\Bigl(2L^{G_0^+}_{\tau_0}(0^+) - \int_0^{\tau_0} \mathbf{1}_Q(Z^\varepsilon_s)\, ds\Bigr) \\
&\qquad + \frac{1}{2}\boldsymbol{E}^d\Bigl(2L^{G_0^+}_{\tau_0} - \int_0^{\tau_0} \mathbf{1}_Q(Z^\varepsilon_s)\, ds\Bigr) \\
&\leqslant (1-\delta) \sup_{r_1, r_2 \in G_0} |u^\varepsilon(r_1) - u^\varepsilon(r_2)| - \frac{1}{2}\boldsymbol{E}^a\Bigl(2L^{G_0^+}_{\tau_0} - \int_0^{\tau_0} \mathbf{1}_Q(Z^\varepsilon_s)\, ds\Bigr) \\
&\qquad + \frac{1}{2}\boldsymbol{E}^d\Bigl(2L^{G_0^+}_{\tau_0} - \int_0^{\tau_0} \mathbf{1}_Q\, ds\Bigr)
\end{aligned} \tag{6.21}$$

Note that by definition of τ_0 we have we have $L^{G_0}_{\tau_0} = 0$ for all $a, d \in A' \cup D'$. Also, if $a \in A'$, then $Y^\varepsilon_s > 0$ for all $s \in [0, \tau_0]$ with probability one. Hence

$$\begin{aligned}
\Bigl| -\frac{1}{2}\boldsymbol{E}^a\Bigl(2L^{G_0^+}_{\tau_0} - \int_0^{\tau_0} \mathbf{1}_Q\, ds\Bigr) + \frac{1}{2}\boldsymbol{E}^d\Bigl(2L^{G_0^+}_{\tau_0} - \int_0^{\tau_0} \mathbf{1}_Q\, ds\Bigr)\Bigr| \\
\leqslant \sup_{d \in D'} \boldsymbol{E}^d \int_0^{\tau_0} \mathbf{1}_Q(Z^\varepsilon_s)\, ds\,.
\end{aligned} \tag{6.22}$$

We claim that the term on the right is bounded by $C\varepsilon^2|\ln\varepsilon|$. To avoid distracting from the main proof, we single this out as a lemma and postpone the proof.

Lemma 6.9. *With the above notation,*

$$\sup_{d \in D'} \boldsymbol{E}^d \int_0^{\tau_0} \mathbf{1}_Q\, ds \leqslant C\varepsilon^2 |\ln\varepsilon|\,.$$

Using Lemma 6.9 and (6.22) in (6.21) we conclude

$$\sup_{a, d \in A' \cup D'} |u^\varepsilon(a) - u^\varepsilon(d)| \leqslant (1-\delta) \sup_{r_1, r_2 \in G_0} |u^\varepsilon(r_1) - u^\varepsilon(r_2)| + C\varepsilon^2|\ln\varepsilon|\,. \tag{6.23}$$

To finish proving (6.18), we will now have to control the oscillation of u^ε on G_0 in terms of the oscillation of u^ε on $A' \cup D'$.

For this, given $Z^\varepsilon_0 \in G_0$, let τ_0' be the first time that Z^ε_t hits $A' \cup D'$. By Itô's formula again, we have for all $r_1, r_2 \in G_0$:

$$\begin{aligned}
u^\varepsilon(r_1) - u^\varepsilon(r_2) &\leqslant \sup_{a', d' \in A' \cup D'} (u^\varepsilon(a') - u^\varepsilon(d')) \\
&\qquad - \frac{1}{2}\boldsymbol{E}^{r_1}\Bigl(2L^{G_0}_{\tau_0'} - \int_0^{\tau_0'} \mathbf{1}_Q\, ds\Bigr) + \frac{1}{2}\boldsymbol{E}^{r_2}\Bigl(2L^{G_0}_{\tau_0'} - \int_0^{\tau_0'} \mathbf{1}_Q\, ds\Bigr)\,.
\end{aligned} \tag{6.24}$$

We claim that the last two terms above are $O(\varepsilon^2)$. For clarity of presentation we single this out as a Lemma and postpone the proof.

Lemma 6.10. *With the above notation*

$$\sup_{r\in G_0}\Big|\boldsymbol{E}^r\Big(2L^{G_0}_{\tau_0'}(0^+)-\int_0^{\tau_0'}\mathbf{1}_Q\,ds\Big)\Big|\leqslant C\varepsilon^2\,.$$

Using (6.24) and Lemma 6.10, we see

$$\sup_{r_1,r_2\in G_0}|u^\varepsilon(r_1)-u^\varepsilon(r_2)|\leqslant\sup_{a,d\in A'\cup D'}|u^\varepsilon(a)-u^\varepsilon(d)|+C\varepsilon^2\,. \tag{6.25}$$

Combining this with (6.23), we obtain

$$\sup_{a,d\in A'\cup D'}|u^\varepsilon(a)-u^\varepsilon(d)|\leqslant(1-\delta)\Big(\sup_{a,d\in A'\cup D'}|u^\varepsilon(a)-u^\varepsilon(d)|+C\varepsilon^2|\ln\varepsilon|\Big)+C\varepsilon^2\,.$$

and hence

$$\sup_{a,d\in A'\cup D'}|u^\varepsilon(a)-u^\varepsilon(d)|\leqslant C\Big(\frac{1-\delta}{\delta}\Big)\varepsilon^2|\ln\varepsilon|+\frac{C}{\delta}\varepsilon^2\,. \tag{6.26}$$

This proves (6.18) as desired.

Now we turn this into an oscillation bound on u^ε over the interior. Observe that for any $z\in\Omega_\varepsilon$,

$$u^\varepsilon(z)=\boldsymbol{E}^z[u^\varepsilon(Z^\varepsilon_{\tau_0'})]+\frac{1}{2}\boldsymbol{E}^z\Big(2L^{Y^\varepsilon}_{\tau_0'}(0^+)-\int_0^{\tau_0'}\mathbf{1}_{\{Y^\varepsilon_s\leqslant 0\}}\,ds\Big) \tag{6.27}$$

These last terms can be estimated with the same argument used in Lemma 6.10, leading to

$$\sup_{z\in\Omega_\varepsilon}|u^\varepsilon(z)-\boldsymbol{E}^z u^\varepsilon(Z^\varepsilon_{\tau_0'})|\leqslant C\varepsilon^2\,.$$

The combination of this and (6.26) implies that

$$\sup_{z_1,z_2\in\Omega_\varepsilon}|u^\varepsilon(z_1)-u^\varepsilon(z_2)|\leqslant\sup_{z_1,z_2\in\Omega_\varepsilon}|\boldsymbol{E}^{z_1}u^\varepsilon(Z^\varepsilon_{\tau_0'})-\boldsymbol{E}^{z_2}u^\varepsilon(Z^\varepsilon_{\tau_0'})|+C\varepsilon^2\leqslant C\varepsilon^2(|\ln\varepsilon|+1)\,.$$

This implies (6.16), concluding the proof. □

For the proof of Lemma 6.8 we use will a standard large deviation estimate for Brownian motion. We state the result we need below.

Lemma 6.11. *Let W_t be a standard Brownian motion in $\mathbb{R}^d$. Let $\gamma\in C([0,T];\mathbb{R}^d)$ be absolutely continuous with $S(\gamma)=\int_0^T|\gamma'(s)|^2\,ds<\infty$. Then*

$$\boldsymbol{P}\left(\sup_{t\in[0,T]}|W(t)-\gamma(t)|\leqslant\delta\right)\geqslant\frac{\boldsymbol{P}(K)}{2}e^{-\frac{1}{2}S(\gamma)-\sqrt{2S(\gamma)/\boldsymbol{P}(K)}}$$

where K is the event $\{\sup_{t\in[0,T]}|W(t)|\leqslant\delta\}$.

The proof of Lemma 6.11 is standard, and can be found in [FW12]. For convenience we provide a proof at the end of this section, and prove Lemmas 6.8, 6.9 and 6.10 next.

Proof of Lemma 6.8. We need to show that for an interval $[r_1, r_2] \subset [-\varepsilon^2/2, \varepsilon^2/2]$,

$$\inf_{z \in A' \cup D'} \boldsymbol{P}^z \left(Z^\varepsilon_{\tau_0} \in [r_1, r_2] \times \{0\} \right) \geqslant C \frac{|r_2 - r_1|}{\varepsilon^2} \,.$$

Suppose $z \in D'$ (the case $z \in A'$ is similar but less complicated by the domain geometry). In order to hit G_0, the process must first hit the boundary of $B(0, \varepsilon^2)$ which is a ball of radius ε^2, centered at the origin $(0,0)$, since $G_0 \subset B(0, \varepsilon^2)$. So, by the strong Markov property, it suffices to show that

$$\inf_{z \in B(0,\varepsilon^2)} \boldsymbol{P}^z \left(Z^\varepsilon_{\tau_0} \in [r_1, r_2] \times \{0\} \right) \geqslant C \frac{|r_2 - r_1|}{\varepsilon^2}.$$

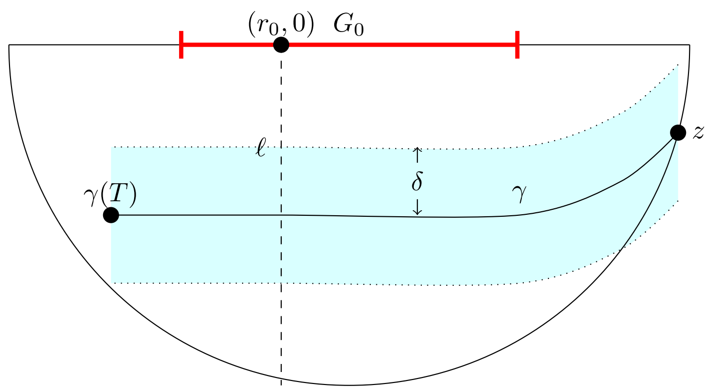

FIGURE 6. The curve γ starts on $\partial B(0, \varepsilon^2)$, goes through the line ℓ while keeping a distance δ from the gate G_0.

Suppose that $[r_1, r_2] = [r_0 - \kappa, r_0 + \kappa]$. Let $\ell = \{r_0\} \times [-\varepsilon^2, 0)$ be the vertical line segment of length ε^2 below the desired exit interval. Let $T = \varepsilon^4$, $\delta = \varepsilon^2/4$, and let γ be a curve parametrized by arc-length such that $\gamma(0) = z$ and the event $\sup_{t \in [0,T]} |Z^\varepsilon(t) - \gamma(t)| \leqslant \delta$ implies that Z^ε hits ℓ before G_0 (see Figure 6). We can choose such a curve γ for which $|\gamma'| \leqslant O(\varepsilon^{-2})$, so that the quantity $S(\gamma)$ in Lemma 6.11 is bounded independent of ε and of $z = \gamma(0) \in B(0, \varepsilon^2)$. Notice also that the set K from Lemma 6.11 satisfies

$$\boldsymbol{P}(K) = \boldsymbol{P}\Big(\sup_{t \in [0,T]} |W(t)| \leqslant \delta \Big) = \boldsymbol{P}\Big(\sup_{t \in [0,1]} |W(t)| \leqslant \frac{\delta}{\sqrt{T}} \Big)$$

by Brownian scaling. Then since $\delta/\sqrt{T}$ is constant, this probability is bounded below and Lemma 6.11 states the probability that Z^ε_t hits ℓ before G_0 is bounded below (away from zero), independent of ε. By the Markov property it now suffices to finish the proof assuming $z_0 \in \ell$. Then consider the unique circle with center at $z_0 \in \ell$ such that the circle intersects G_0 at the points $(r_0 - \kappa, 0)$ and $(r_0 + \kappa, 0)$. By symmetry of Brownian motion, the exit distribution on the circle is uniform. The probability that $Z^\varepsilon_{\tau_0} \in [r_0 - \kappa, r_0 + \kappa]$ is at least the probability of exiting this circle along the arc above G_0, which is the ratio of the arc length to the circumference. This probability is bounded below by $2\kappa/(\varepsilon^2) \gtrsim |r_1 - r_2|/(\varepsilon^2)$. $\square$

Proof of Lemma 6.9. By the Markov property, Lemma 6.9 will follow from the estimate

$$\sup_{z\in Q} \boldsymbol{E}^z \int_0^{\tau_0} \mathbf{1}_Q(Z^\varepsilon_s)\, ds \leqslant C\varepsilon^2 |\ln \varepsilon|\,. \tag{6.28}$$

Let $D = \{\pm\varepsilon/2\} \times [-\varepsilon, 0]$ be the sides of Q, and recall D' (defined in (6.17)) denotes the sides of Q'. We consider two sequences of stopping times, ζ_i, η_i, denoting successive visits of of Z^ε to $G_0 \cup D'$ and D respectively. Precisely, let $\eta_0 = 0$, inductively define

$$\zeta_i = \inf\{s > \eta_{i-1} \mid Z^\varepsilon_s \in G_0 \cup D'\}$$
$$\eta_i = \inf\{s > \zeta_i \mid Z^\varepsilon_s \in D\}\,,$$

for $i \in \{1, 2, \dots\}$, and let

$$M = \min\{n \in \mathbb{N} \mid Z^\varepsilon_{\zeta_n} \in G_0\}\,.$$

Notice that $\zeta_M = \tau_0$. Using the strong Markov property, and the fact that $Z^\varepsilon_s \notin Q$ for $s \in (\zeta_i, \eta_i)$ for all $i < M$, we obtain

$$\boldsymbol{E}^z \int_0^{\tau_0} \mathbf{1}_Q(Z^\varepsilon_s)\, ds = \boldsymbol{E}^z \sum_{i=1}^{M} \int_{\eta_{i-1}}^{\zeta_i} \mathbf{1}_Q(Z^\varepsilon_s)\, ds = \boldsymbol{E}^z \sum_{i=1}^{M} \boldsymbol{E}^{Z^\varepsilon_{\eta_{i-1}}} \int_0^{\zeta_1} \mathbf{1}_Q(Z^\varepsilon_s)\, ds$$
$$\leqslant (\boldsymbol{E}^z M)\Big(\sup_{d\in D} \boldsymbol{E}^d \int_0^{\zeta_1} \mathbf{1}_Q(Z^\varepsilon_s)\, ds\Big)\,. \tag{6.29}$$

Since ζ_1 is bounded by the exit time of a one dimensional Brownian motion (the first coordinate of Z^ε) from an interval of length $3\varepsilon/2$, we know

$$\sup_{d\in D} \boldsymbol{E}^d \zeta_1 \leqslant C\varepsilon^2\,.$$

Using this in (6.29) shows

$$\boldsymbol{E}^z \int_0^{\tau_0} \mathbf{1}_Q(Z^\varepsilon_s)\, ds \leqslant C\varepsilon^2 \boldsymbol{E}^z M\,. \tag{6.30}$$

We now estimate $\boldsymbol{E}^z M$. Notice that

$$\begin{aligned}
\boldsymbol{P}^z(M \geqslant n) &= \boldsymbol{P}^z(Z^\varepsilon_{\zeta_1} \notin G_0,\ Z^\varepsilon_{\zeta_2} \notin G_0,\ \dots,\ Z^\varepsilon_{\zeta_n} \notin G_0)\\
&= \boldsymbol{E}^z\Big(\mathbf{1}_{\{Z^\varepsilon_{\zeta_1}\notin G_0,\ Z^\varepsilon_{\zeta_2}\notin G_0,\ \dots,\ Z^\varepsilon_{\zeta_{n-1}}\notin G_0\}} \boldsymbol{P}^{Z^\varepsilon_{\eta_{n-1}}}(Z^\varepsilon_{\zeta_1} \notin G_0)\Big)\\
&\leqslant \boldsymbol{P}^z(Z^\varepsilon_{\zeta_1} \notin G_0,\ Z^\varepsilon_{\zeta_2} \notin G_0,\ \dots,\ Z^\varepsilon_{\zeta_{n-1}} \notin G_0)\Big(\sup_{d\in D} \boldsymbol{P}^d(Z^\varepsilon_{\zeta_1} \notin G_0)\Big)\\
&= \boldsymbol{P}^z(M \geqslant n-1)\Big(\sup_{d\in D} \boldsymbol{P}^d(Z^\varepsilon_{\zeta_1} \notin G_0)\Big)\,.
\end{aligned}$$

Thus, by induction

$$\boldsymbol{P}^z(M \geqslant n) \leqslant \Big(\sup_{d\in D} \boldsymbol{P}^d(Z^\varepsilon_{\zeta_1} \notin G_0)\Big)^n\,.$$

Now we claim that there exist a constant $c_0 > 0$, independent of ε, such that

$$\sup_{d\in D} \boldsymbol{P}^d\big(Z^\varepsilon_{\zeta_1} \notin G_0\big) < 1 - \frac{c_0}{|\ln \varepsilon|}\,. \tag{6.31}$$

This is the key step in the proof. Once established, it implies

$$\boldsymbol{E}^z M = \sum_{n=1}^{\infty} \boldsymbol{P}^z(M \geqslant n) \leqslant \sum_{n=1}^{\infty} \Big(1 - \frac{c_0}{|\ln \varepsilon|}\Big)^{n-1} = \frac{|\ln \varepsilon|}{c_0}\,,$$

which when combined with with (6.30) yields

$$\sup_{z \in D} \boldsymbol{E}^z \int_0^{\tau_1} \mathbf{1}_Q(Z_s^\varepsilon)\, ds \leqslant \frac{C\varepsilon^2 |\ln \varepsilon|}{c_0}\,. \tag{6.32}$$

This proves (6.28) and finishes the proof of Lemma 6.9.

Thus it only remains to prove (6.31). We will prove it by showing

$$\inf_{z \in D} \boldsymbol{P}^z\big(Z_{\zeta_1}^\varepsilon \in G_0\big) > \frac{c_0}{|\ln \varepsilon|}\,. \tag{6.33}$$

We will prove this in three stages. First, by scaling, it is easy to see that the probability that starting from D the process Z^ε hits $B(0, \varepsilon/4)$ before D' with probability $c_0 > 0$. Next, using the explicit Greens function in an annulus we show that the probability that starting from $B(0, \varepsilon/4)$, the process Z^ε hits $B(0, \varepsilon^2)$ before exiting $B(0, \varepsilon/2)$ with probability $c_0/|\ln \varepsilon|$. Finally, by scaling, it again follows that that starting from $B(0, \varepsilon^2)$ the process Z^ε hits G_0 before exiting $B(0, 2\varepsilon^2)$ with probability $c_0 > 0$.

For the first stage, consider the stopping times

$$\sigma_{\varepsilon/4} = \inf\Big\{t > 0 \;\Big|\; Z_t^\varepsilon \in B\Big(0, \frac{\varepsilon}{4}\Big)\Big\}\,,$$
$$\sigma_{D'} = \inf\{t > 0 \mid Z_t^\varepsilon \in D'\}\,.$$

By rescaling, it immediately follows that

$$\inf_{z \in D} \boldsymbol{P}(\sigma_{\varepsilon/4} < \sigma_{D'} \mid Z_0^\varepsilon = z) \geqslant p_1\,, \tag{6.34}$$

for some $p_1 > 0$, independent of ε.

For the second stage suppose for $Z_0^\varepsilon \in \partial B(0, \varepsilon/4)$. Consider the stopping times σ_{ε^2} and $\sigma_{\varepsilon/2}$ defined by

$$\sigma_{\varepsilon^2} = \inf\{t > 0 \mid Z_t^\varepsilon \in \partial B(0, \varepsilon^2)\}\,,$$
$$\sigma_{\varepsilon/2} = \inf\{t > 0 \mid Z_t^\varepsilon \in \partial B(0, \varepsilon/2)\}\,.$$

The function

$$f(z) = \frac{\ln(2|z|/\varepsilon)}{\ln(2\varepsilon)}$$

is harmonic in $B(0, \varepsilon/2) \setminus B(0, \varepsilon^2)$ and satisfies $f = 1$ on $\partial B(0, \varepsilon^2)$, and $f = 0$ on $\partial B(0, \varepsilon/2)$. This implies that for all $z \in B(0, \varepsilon/4)$ we have

$$\boldsymbol{P}^z(\sigma_{\varepsilon^2} < \sigma_{\varepsilon/2}) = f(z) = \frac{\ln(1/2)}{\ln(2\varepsilon)}\,. \tag{6.35}$$

Finally, for the last stage, let $\sigma_{2\varepsilon^2}$ be the first time Z^ε exits $B(0, 2\varepsilon^2)$. By scaling, it immediately follows that for all $z \in \partial B(0, \varepsilon^2)$

$$\boldsymbol{P}^z(\tau_0 < \sigma_{2\varepsilon^2}) \geqslant p_2\,, \tag{6.36}$$

for some constant $p_2 > 0$, independent of ε.

The strong Markov property and (6.34), (6.35), and (6.36) imply

$$\inf_{z\in D} \boldsymbol{P}(Z^\varepsilon_{\zeta_1} \in G_0 \mid Z^\varepsilon_0 = z) \geqslant p_1 \cdot \frac{\ln(1/2)}{\ln(2\varepsilon)} \cdot p_2\,.$$

By the time-homogeneity of the Markov process Z^ε, this establishes (6.33), finishing the proof. □

Proof of Lemma 6.10. To estimate the local time term, consider the function

$$w(x,y) = \begin{cases} \varepsilon^2 - y\,, & y \in [0,\varepsilon^2]\,,\\ \varepsilon^2, & \text{otherwise}, \end{cases}$$

which satisfies $\partial_y w(x,0^+) - \partial_y w(x,0^-) = -1$ for $x \in [-\varepsilon^2/2, \varepsilon^2/2]$. Let $\tau_{A'}$ be the first hitting time to the set A', where we know $w = 0$. Using Itô's formula we obtain

$$\boldsymbol{E}^z L^{G_0}_{\tau_{A'}} = w(z), \quad z \in G_0.$$

Clearly $\tau_{A'} \geqslant \tau_0'$, and so

$$\sup_{z\in G_0} \boldsymbol{E}^z L^{G_0}_{\tau_0'} \leqslant \sup_{z\in G_0} \boldsymbol{E}^z L^{G_0}_{\tau_{A'}} = \varepsilon^2\,.$$

Next, we estimate the term

$$\sup_{z\in G_0} \boldsymbol{E}^z \int_0^{\tau_0'} \mathbf{1}_Q(Z^\varepsilon_s)\,ds\,. \tag{6.37}$$

Let $\tau_{D'} = \inf\{t > 0 \mid Z^\varepsilon_t \in D'\}$, so that $\tau_{D'} \geqslant \tau_0'$. Let $H = \{(x,y) \in \mathbb{R}^2 \mid y = -\varepsilon\}$ denote the bottom boundary of Ω, and let $H' = [-3\varepsilon/4, 3\varepsilon/4] \times \{-\varepsilon\} = \bar{Q}' \cap H$. We now consider repeated visits to H' before hitting D'. For this, define the stopping times $\{\zeta_k\}_{k=0}^\infty$ inductively by

$$\begin{aligned} \zeta_0 &= \inf\{t > 0 \mid Z^\varepsilon_t \in H\}\,,\\ \zeta_k &= \inf\{t \geqslant \zeta_{k-1} + \varepsilon^2 \mid Z^\varepsilon_t \in H\}\,, \quad \text{for } k = 1,2,3,\dots\,, \end{aligned}$$

and define

$$M = \min\{k \in \mathbb{N} \mid Z^\varepsilon_{\zeta_k} \in H \setminus H'\}\,.$$

Observe that if $Z^\varepsilon_0 \in G_0$, then $\tau_{D'} \leqslant \zeta_M$. Indeed, since $Z^\varepsilon_{\zeta_M} \in H \setminus H'$ and trajectories of process Z^ε is continuous, they must must have passed through the set D' at some time before ζ_M.

Now, to bound (6.37) we observe

$$\int_0^{\tau_0'} \mathbf{1}_Q(Z^\varepsilon_s)\,ds \leqslant \int_0^{\zeta_0} \mathbf{1}_Q(Z^\varepsilon_s)\,ds + \sum_{k=1}^M \int_{\zeta_{k-1}}^{\zeta_k} \mathbf{1}_Q(Z^\varepsilon_s)\,ds. \tag{6.38}$$

On the event $\{M > k-1\}$ we must have $Z^\varepsilon_{\zeta_{k-1}} \in H'$. Using this observation, the strong Markov property, and the time-homogeneity of the process, we see that for any $z \in G_0$ we have

$$\begin{aligned} \boldsymbol{E}^z \int_0^{\tau_0'} \mathbf{1}_Q(Z^\varepsilon_s)\,ds &\leqslant \boldsymbol{E}^z \int_0^{\zeta_0} \mathbf{1}_Q(Z^\varepsilon_s)\,ds + \boldsymbol{E}^z \sum_{k=1}^M \int_{\zeta_{k-1}}^{\zeta_k} \mathbf{1}_Q(Z^\varepsilon_s)\,ds\\ &= \boldsymbol{E}^z \int_0^{\zeta_0} \mathbf{1}_Q(Z^\varepsilon_s)\,ds + \boldsymbol{E}^z \sum_{k=1}^M \boldsymbol{E}^{Z^\varepsilon_{\zeta_{k-1}}} \int_{\zeta_0}^{\zeta_1} \mathbf{1}_Q(Z^\varepsilon_s)\,ds \end{aligned}$$

$$\leqslant \boldsymbol{E}^z \int_0^{\zeta_0} \mathbf{1}_Q(Z_s^\varepsilon)\,ds + \boldsymbol{E}^z \sum_{k=1}^{M} \sup_{z'\in H'} \boldsymbol{E}^{z'} \int_{\zeta_0}^{\zeta_1} \mathbf{1}_Q(Z_s^\varepsilon)\,ds$$

$$(6.39)\qquad = \boldsymbol{E}^z \int_0^{\zeta_0} \mathbf{1}_Q(Z_s^\varepsilon)\,ds + (\boldsymbol{E}^z M) \sup_{z'\in H'} \boldsymbol{E}^{z'} \int_{\zeta_0}^{\zeta_1} \mathbf{1}_Q(Z_s^\varepsilon)\,ds\,.$$

We now bound the right hand side of (6.39). Note

$$(6.40)\quad \boldsymbol{E}^z M = \sum_{j=1}^{\infty} \boldsymbol{P}^z(M \geqslant j) = \sum_{j=1}^{\infty} \boldsymbol{P}^z(Z_{\zeta_0}^\varepsilon \in H',\ Z_{\zeta_1}^\varepsilon \in H',\ \dots,\ Z_{\zeta_{j-1}}^\varepsilon \in H')\,.$$

By the Markov property

$$\boldsymbol{P}^z\big(Z_{\zeta_{i+1}}^\varepsilon \in H',\ Z_{\zeta_i}^\varepsilon \in H'\big) = \boldsymbol{E}^z\big(\mathbf{1}_{Z_{\zeta_i}^\varepsilon \in H'} \boldsymbol{P}^{Z_{\zeta_i}^\varepsilon}(Z_{\zeta_1}^\varepsilon \in H')\big)$$

$$(6.41)\qquad \leqslant \Big(\sup_{z'\in H'} \boldsymbol{P}^{z'}(Z_{\zeta_1}^\varepsilon \in H')\Big) \boldsymbol{P}^z(Z_{\zeta_i}^\varepsilon \in H')$$

Now using Lemma 6.11 and the fact that $\zeta_1 \geqslant \varepsilon^2$, one can show that

$$\sup_{z'\in H'} \boldsymbol{P}^{z'}(Z_{\zeta_1}^\varepsilon \in H') \leqslant 1 - c_0\,,$$

for some constant $c_0 > 0$, independent of ε. Combining this with (6.41) and using induction we obtain

$$\sum_{j=1}^{\infty} \boldsymbol{P}^z(Z_{\zeta_0}^\varepsilon \in H',\ Z_{\zeta_1}^\varepsilon \in H',\ \dots,\ Z_{\zeta_{j-1}}^\varepsilon \in H') \leqslant \sum_{j=1}^{\infty} (1-c_0)^{j-1}\,.$$

Thus, using (6.40) we see

$$\boldsymbol{E}^z M \leqslant \frac{1}{c_0}\,.$$

Using this in (6.39) we have

$$\boldsymbol{E}^z \int_0^{\tau_0'} \mathbf{1}_Q(Z_s^\varepsilon)\,ds \leqslant \boldsymbol{E}^z \int_0^{\zeta_0} \mathbf{1}_Q(Z_s^\varepsilon)\,ds + \frac{1}{c_0} \sup_{z'\in H'} \boldsymbol{E}^{z'} \int_{\zeta_0}^{\zeta_1} \mathbf{1}_Q(Z_s^\varepsilon)\,ds$$

$$(6.42)\qquad \leqslant \boldsymbol{E}^z \int_0^{\zeta_0} \mathbf{1}_Q(Z_s^\varepsilon)\,ds + \frac{1}{c_0}\Big(\varepsilon^2 + \sup_{z'\in\Omega} \boldsymbol{E}^{z'} \int_0^{\zeta_0} \mathbf{1}_Q(Z_s^\varepsilon)\,ds\Big)\,.$$

To bound this, consider the function

$$v(x,y) = \begin{cases} \frac{1}{2}(\varepsilon^2 - y^2)\,, & y \in [-\varepsilon, 0]\,, \\ \frac{1}{2}\varepsilon^2\,, & y > 0\,. \end{cases}$$

and observe that for any $z \in \Omega_\varepsilon$,

$$\boldsymbol{E}^z \int_0^{\zeta_0} \mathbf{1}_Q(Z_s^\varepsilon)\,ds \leqslant \boldsymbol{E}^z \zeta_0 = v(z) \leqslant \frac{\varepsilon^2}{2}\,.$$

Substituting this in (6.42) shows

$$\boldsymbol{E}^z \int_0^{\tau_0'} \mathbf{1}_Q(Z_s^\varepsilon)\,ds \leqslant \Big(\frac{1}{2} + \frac{3}{2c_0}\Big)\varepsilon^2\,,$$

completing the proof. □

Finally, for completeness we prove Lemma 6.11. The proof is a standard argument using the Girsanov theorem, and can for instance be found in [FW12].

Proof of Lemma 6.11. Define $Y(t) = W(t) - \gamma(t)$. Let $B(t)$ be an independent Brownian motion in $\mathbb{R}$ with respect to measure $\boldsymbol{P}$. Let define a new measure $\mathcal{Q}$ by

$$\frac{d\mathcal{Q}}{d\boldsymbol{P}} = e^{-\int_0^T \gamma'(s)\,dB(s) - \frac{1}{2}\int_0^T |\gamma'(s)|^2\,ds}$$

Let $\tilde{K}$ be the event $\tilde{K} = \tilde{K}_{T,\delta} = \{\sup_{t\in[0,T]} |B(t)| \leqslant \delta\}$. Let $S(\gamma) = \int_0^T |\gamma'(s)|^2\,ds$. According to the Girsanov theorem,

$$\begin{aligned}\boldsymbol{P}(\sup_{t\in[0,T]} |Y(t)| \leqslant \delta) &= \mathcal{Q}(\tilde{K}) \\ &= \boldsymbol{E}^{\boldsymbol{P}}\left[\mathbf{1}_{\tilde{K}} e^{-\int_0^T \gamma'(s)\,dB(s) - \frac{1}{2}\int_0^T |\gamma'(s)|^2\,ds}\right] \\ &= e^{-\frac{1}{2}S(\gamma)}\boldsymbol{E}^{\boldsymbol{P}}\left[\mathbf{1}_{\tilde{K}} e^{-\int_0^T \gamma'(s)\,dB(s)}\right]\end{aligned}$$

Now, by Chebychev and the Itô isometry,

$$\boldsymbol{P}\left(\int_0^T \gamma'(s)\,dB(s) \geqslant \alpha\sqrt{S(\gamma)}\right) \leqslant \frac{1}{\alpha^2}$$

So, if $\frac{1}{\alpha^2} \leqslant \frac{1}{2}\boldsymbol{P}(\tilde{K})$, we have

$$\boldsymbol{P}\left(\sup_{t\in[0,T]} |Y(t)| \leqslant \delta\right) \geqslant e^{-\frac{1}{2}S(\gamma) - \alpha\sqrt{S(\gamma)}})\frac{1}{2}\boldsymbol{P}(\tilde{K})$$

In particular, by choosing $\alpha = \sqrt{2/\boldsymbol{P}(\tilde{K})} > 0$, we have

$$\boldsymbol{P}\left(\sup_{t\in[0,T]} |Y(t)| \leqslant \delta\right) \geqslant e^{-\frac{1}{2}S(\gamma) - \sqrt{2S(\gamma)/\boldsymbol{P}(\tilde{K})}}\frac{1}{2}\boldsymbol{P}(\tilde{K})$$

Note: $\boldsymbol{P}(\tilde{K}) = \boldsymbol{P}(K)$ since B and W have the same law under $\boldsymbol{P}$. □

6.5. **Local time on teeth boundaries (Lemma 6.6).** The last remaining lemma to prove is Lemma 6.6 which is the local time balance within the teeth. We again use the symmetry and geometric series arguments as in the proof of Proposition 6.7.

Proof of Lemma 6.6. As with (6.6), we will estimate

$$(6.43)\quad I_k \stackrel{\text{def}}{=} \boldsymbol{E}^z\Big(\int_0^t \frac{1}{2}\partial_x^2 f(Z_s^\varepsilon)\mathbf{1}_{\{Y_s^\varepsilon > 0\}}\mathbf{1}_{\{|X_s^\varepsilon - \varepsilon k| < \varepsilon/2\}}\,ds + \int_0^t \partial_x f(Z_s^\varepsilon)\mathbf{1}_{\{|X_s^\varepsilon - \varepsilon k| < \varepsilon/2\}}\,dL_s^{\pm}\Big)$$

for any $z \in K \cap \Omega_\varepsilon$. As before, Lemma 6.6 will follow if we can show that for any finite M, $\sum_{\varepsilon|k|<M} I_k$ vanishes as $\varepsilon \to 0$. Since there are $O(1/\varepsilon)$ terms in the sum, it suffices to bound each I_k by $o(\varepsilon)$. Without loss of generality, assume $k = 0$ and let $T_0 = [-\alpha\varepsilon^2/2, \alpha\varepsilon^2/2] \times [0,1]$ denote the tooth centered at $k = 0$. Define the function $\tilde{f}\colon T_0 \to \mathbb{R}$ by

$$\tilde{f}(x,y) \stackrel{\text{def}}{=} f(x,y) - f(0,y) - x\partial_x f(0,y)\,,$$

Note that for all $(x,y) \in T_0$ we have

$$\tilde{f}(0,y) = 0\,,\qquad \partial_x\tilde{f}(0,y) = 0\,,\qquad\text{and}\qquad \partial_x^2\tilde{f}(x,y) = \partial_x^2 f(x,y)\,.$$

and hence $\|\tilde{f}\|_\infty = O(\varepsilon^4)$. Moreover,

$$\partial_y^2 \tilde{f}(x,y) = \partial_y^2 f(x,y) - \partial_y^2 f(0,y) - x\partial_x \partial_y^2 f(0,y) = O(\varepsilon^4),$$

assuming $\partial_y^2 f \in C^1$, and $\partial_y \tilde{f}(x,0) = O(\varepsilon^2)$ for $x \in [-\alpha\varepsilon^2/2, \alpha\varepsilon^2/2]$.

We now extend the definition of $\tilde{f}$ continuously outside of T_0 (into the spine) to a $O(\varepsilon^2)$ neighborhood of G as follows. Let $\eta(x,y)$ be a smooth, radially-symmetric cutoff function, vanishing outside of $B_2(0,0)$ and such that $\eta(z) = 1$ for $|z| \leqslant 1$. Then, for $y \leqslant 0$ (i.e. outside the tooth T_0), define

$$\tilde{f}(x,y) \stackrel{\text{def}}{=} \eta\Big(\frac{x}{\alpha\varepsilon^2}, \frac{y}{\alpha\varepsilon^2}\Big)\Big(f(x,0) - f(0,0) - x\partial_x f(0,0)\Big).$$

In this way, $\tilde{f}$ has the additional properties that

(1) $\tilde{f}$ vanishes outside of $T_0 \cup B_{2\alpha\varepsilon^2}(0,0)$,
(2) $\partial_y \tilde{f} = 0$ on $(\partial Q) \setminus G$,
(3) The jump in $\partial_y \tilde{f}$ across G is $O(\varepsilon^4)$.
(4) $\Delta \tilde{f} = O(1)$ in the region $B^-_{2\alpha\varepsilon^2} = \{y \leqslant 0\} \cap B_{2\alpha\varepsilon^2}(0,0)$.

This last point stems from the fact that $|f(x,0) - f(0,0) - x\partial_x f(0,0)| = O(\varepsilon^4)$. In view of this construction, we see that

$$\begin{aligned} I_0 = \boldsymbol{E}^z\Big(\int_0^t \frac{1}{2}(\partial_x^2 \tilde{f} + \partial_y^2 \tilde{f})(Z_s^\varepsilon)\mathbf{1}_{\{Z_s^\varepsilon \in T_0\}}\, ds - \int_0^t \partial_x \tilde{f}(Z_s^\varepsilon)\mathbf{1}_{\{Z_s^\varepsilon \in T_0\}} dL_s^+\Big) \\ + \boldsymbol{E}^z\Big(\int_0^t \partial_x f(0, Y_s^\varepsilon) d(L_s^- - L_s^+)\Big) + O(\varepsilon^2)t \\ = R_1 + R_2 + O(\varepsilon^2)t\,. \end{aligned}$$

Notice how we have introduced the $\partial_y^2 \tilde{f}$ term for the price of $O(\varepsilon^2)t$. We also still have $\partial_y \tilde{f}(x,1) = 0$ on the top boundary of the tooth. By Itô's formula applied to $\tilde{f}$, we have

$$\begin{aligned} R_1 &= \boldsymbol{E}^z[\tilde{f}(Z_t^\varepsilon) - \tilde{f}(Z_0^\varepsilon)] + \boldsymbol{E}^z\Big(\int_0^t \partial_y \tilde{f}(X_s^\varepsilon, 0) dL^G\Big) + \boldsymbol{E}^z\Big(\int_0^t O(1)\mathbf{1}_{B_{2\alpha\varepsilon^2}}(Z_s)\, ds\Big) \\ &= O(\varepsilon^4) + O(\varepsilon^2)\boldsymbol{E}^z\Big(L_t^G\Big) + O(1)\boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{B^-_{2\alpha\varepsilon^2}}(Z_s)\, ds\Big) \\ &= O(\varepsilon^4) + O(\varepsilon^2) + O(1)R_3, \end{aligned}$$

by since $\boldsymbol{E}^z L_t^G = O(1)$ by (6.11).

We now estimate the term R_2. By symmetry with respect to reflection in the y coordinate, we note that

$$\boldsymbol{E}^{z'}\Big(\int_0^t \partial_x f(0, Y_s^\varepsilon) d(L_s^- - L_s^+)\Big) = 0$$

for any $z' = (0,y)$ on the axis of the tooth T_0. Thus by symmetry and the Markov property, it suffices to estimate

$$\boldsymbol{E}^z\Big(\int_0^\tau \partial_x f(0, Y_s^\varepsilon)\, dL_s^+\Big),$$

where $\tau = \inf\{t \mid X_t^\varepsilon = 0\}$ is the first time that Z_t^ε reaches this x-axis $\{0\} \times \mathbb{R}$, and z is to the right of the y-axis. Clearly this is bounded by $\|\partial_x f\|_\infty \boldsymbol{E}^z L_\tau^+$. Moreover, using $x \wedge \alpha\varepsilon^2/2$ as a test function, we immediately see $\boldsymbol{E}^z L_\tau^+ \leqslant \alpha\varepsilon^2/2$. This shows $R_2 = O(\varepsilon^2)$ as desired.

Finally, we estimate the term

$$R_3 = \boldsymbol{E}^z\Big(\int_0^t \mathbf{1}_{B^-_{2\alpha\varepsilon^2}}(Z_s)\,ds\Big),$$

where $B^-_{2\alpha\varepsilon^2} = \{y \leqslant 0\} \cap B_{2\varepsilon^2}(0,0)$. The geometry of the domain Ω_ε makes this estimate a little tedious. Since the proof is very similar to the arguments used in the proof of Proposition 6.7, we do not spell out all the details here.

We will show that $R_3 \leqslant O(\varepsilon^3|\log(\varepsilon)|)$. For this, we first claim

$$\sup_{z\in\Omega_\varepsilon\cap K} \boldsymbol{E}^z\Big(\int_0^{\tau_{4\alpha\varepsilon^2}} \mathbf{1}_{B^-_{2\alpha\varepsilon^2}}(Z_s)\,ds\Big) \leqslant O(\varepsilon^4)$$

where $\tau_{4\alpha\varepsilon^2} = \inf\{t \mid Z^\varepsilon_t \in D^-_{4\varepsilon^2}\}$, and $D^-_{4\alpha\varepsilon^2} = \{y \leqslant 0\} \cap \partial B_{4\alpha\varepsilon^2}(0,0)$. This follows by directly applying Itô's formula with a function f satisfying $\Delta f \leqslant 0$ in $\{y \leqslant 0\} \cap B_{4\alpha\varepsilon^2}(0,0)\}$, with $\Delta f \leqslant -c < 0$ in $B^-_{2\alpha\varepsilon^2}$.

Next, we claim that there is $C > 0$ such that

$$\inf_{z\in D^-_{4\alpha\varepsilon^2}} \boldsymbol{P}^z\Big(\sigma_{5\varepsilon} \leqslant \tau_{2\alpha\varepsilon^2}\Big) \geqslant \frac{C}{|\log(\varepsilon)|}\,,$$

where $\sigma_{5\varepsilon} = \inf\{t \mid |X^\varepsilon_t| = 5\varepsilon\}$ and $\tau_{2\varepsilon^2} = \inf\{t \mid Z^\varepsilon_t \in B^-_{2\alpha\varepsilon^2}\}$. This is the narrow escape asymptotics [HS14], and follows from a direct calculation with the Greens function in a manner similar to the proof of (6.31). Finally, we claim that for any $t > 0$, there is $C > 0$ such that

$$\inf_{\{|x|=5\varepsilon\}} \boldsymbol{P}^z(\tau_{2\alpha\varepsilon^2} \geqslant t) \geqslant C\varepsilon.$$

This follows from comparison between X^ε_t and a standard Brownian motion on $\mathbb{R}$, via Lemma 6.4. Thus, starting from $z \in D^-_{4\alpha\varepsilon^2}$, with probability at least $C\varepsilon/|\log(\varepsilon)|$ the process Z_t will make a long excursion such that it doesn't return to $B^-_{2\alpha\varepsilon^2}$ before time t. Using the same geometric series argument as in the proof of Lemma 6.9, we have

$$R_3 \leqslant C(\log(\varepsilon)/\varepsilon) \sup_z \boldsymbol{E}^z\Big(\int_0^{\tau_{4\alpha\varepsilon^2}} \mathbf{1}_{B^-_{2\alpha\varepsilon^2}}(Z_s)\,ds\Big) = O(\varepsilon^3|\log(\varepsilon)|)\,,$$

as claimed.

Finally, combining all these estimates we conclude that for any k, I_k (defined in (6.43)) is at most $O(\varepsilon^2)$. Consequently $\sum_{\varepsilon|k|<M} I_k \to 0$ as $\varepsilon \to 0$, concluding the proof. ☐

7. Future Work

7.1. Other Scaling Limits for the Fat Comb. The fat comb also potentially admits other scaling limits by choosing different values for the width of the spine and the teeth. Consider for instance the case where

$$\Omega_\varepsilon = \{(x,y) \in \mathbb{R}^2 \mid -1 < y < \mathbf{1}_{B(\varepsilon\mathbb{Z},\varepsilon/2)}(x)\}\,, \tag{7.1}$$

Conjecture 7.1. *Suppose Z^ε is the reflected diffusion in the above domain Ω_ε. Then, $Z^\varepsilon \to (X,Y)$ in law as $\varepsilon \to 0$, where:*

(1) *The process Y is a doubly reflected Brownian motion on $(-1,1)$.*

(2) The process X satisfies the SDE

$$dX_t = \mathbf{1}_{\{Y_t \leqslant 0\}}\, d\bar{W}_t\,,$$

for a Brownian motion $\bar{W}$ that is independent of Y.

Remark 7.2. Let A be the generator of the process (X, Y) in Conjecture 7.1. Then

$$Af = \begin{cases} \Delta f & y \leqslant 0\,, \\ \partial_y^2 f & y > 0\,. \end{cases}$$

with $\mathcal{D}(A)$ chosen so that $Af \in C^0$. Then either $f \notin C^2$ or $\partial_x^2 f(x, 0) = 0$ is a necessary condition. It's unclear at the moment which is the correct condition to enforce.

7.2. **Clark's Model.** It still remains to prove the homogenization of the diffusion associated to the Clark Model from §1.3 which we recall here. Let B be an open set with Lipschitz boundary such that $\bar{B} \subset (0,1)^2$, which we then extend periodically to all of $\mathbb{R}^2$. Let $F = \mathbb{R}^2 \backslash \bar{B}$. Consider the divergence form parabolic equation: let Ω_B be an open set with Lipschitz boundary such that $\bar{\Omega}_B \subset \mathbb{T}^2$. We denote by B the periodic extension to all of $\mathbb{R}^2$. Let $F = \mathbb{R}^2 \backslash \bar{B}$.

$$\partial_t u^\varepsilon - \nabla \cdot \big(a^\varepsilon \nabla u^\varepsilon\big) = f\,, \tag{7.2}$$

$$u_0^\varepsilon(x) = \mathbf{1}_F\Big(\frac{x}{\varepsilon}\Big) u_0(x) + \mathbf{1}_B\Big(\frac{x}{\varepsilon}\Big) U_0\Big(x, \frac{x}{\varepsilon}\Big)\,. \tag{7.3}$$

The diffusivity a^ε is given by

$$a^\varepsilon(x) = \mathbf{1}_F\Big(\frac{x}{\varepsilon}\Big) a_F\Big(\frac{x}{\varepsilon}\Big) + \varepsilon^2 \mathbf{1}_B\Big(\frac{x}{\varepsilon}\Big) a_B\Big(\frac{x}{\varepsilon}\Big),$$

where $a_F \in C^1(\bar{F}; \mathbb{R}^{d\times d}_{\mathrm{Sym}})$, $a_B \in C^1(\bar{B}; \mathbb{R}^{d\times d}_{\mathrm{Sym}})$ such that $a_F, a_B \geqslant \alpha I$ for some $\alpha > 0$.

Let Z^ε be the diffusion on associated with generator

$$\nabla \cdot a^\varepsilon \nabla$$

in $\mathbb{R}^2$. The SDE describing Z^ε is

$$dZ_t^\varepsilon = \sigma^\varepsilon\, dW_t + \nabla \cdot a^\varepsilon\, dt\, + q^\varepsilon\, d\ell_t^\varepsilon$$

where σ^ε is a matrix such that $a^\varepsilon = \frac{1}{2}\sigma\sigma^*$ and

$$(\nabla \cdot a^\varepsilon)_j = \partial_i a^\varepsilon_{i,j}\,.$$

Above $\ell_t^\varepsilon = L^{Z^\varepsilon}(\partial B^\varepsilon)$ is the local time of Z^ε on the boundary of the rescaled blocks ∂B^ε where $B^\varepsilon = \varepsilon B$. To write q^ε explicitly we define $\sigma_F^v = \sqrt{a_F v \cdot v}$ and $\sigma_B^v = \sqrt{a_B v \cdot v}$ for $v \in \mathbb{R}^2$ which is the diffusion coefficient in the direction v. Then $q^\varepsilon \colon \partial B^\varepsilon \to \mathbb{R}^2$ is given by

$$q^\varepsilon(x) = \frac{\sigma_F^\nu\big(\frac{x}{\varepsilon}\big) - \varepsilon \sigma_B^\nu\big(\frac{x}{\varepsilon}\big)}{\sigma_F^\nu\big(\frac{x}{\varepsilon}\big) + \varepsilon \sigma_B^\nu\big(\frac{x}{\varepsilon}\big)} \nu\Big(\frac{x}{\varepsilon}\Big)\,,$$

where ν is the outward pointing unit normal on ∂B.

Let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ denote the projection of $\mathbb{R}^2$ onto the torus, and let $Y^\varepsilon = \pi(Z^\varepsilon/\varepsilon)$ if $Z^\varepsilon \in B^\varepsilon$ and extend it right continuously if $Z^\varepsilon/\varepsilon \notin B$.

Conjecture 7.3. *As $\varepsilon \to 0$ the pair of processes $(X^\varepsilon, Y^\varepsilon)$ converge weakly to a process (X, Y) described below.*

We begin by describing the limiting process Y, which can heuristically be described as follows: In the interior of Ω_B, Y is simply an Itô diffusion with generator $\nabla \cdot a_B \nabla$. When Y hits the boundary of B, it performs a random jump to another point on $\partial\Omega_B$ and is reflected with some stickyness factor. A rigorous construction of this process can be obtained abstractly through the Hille-Yosida theorem. Let $A_Y = \nabla \cdot a_B \nabla$, and define $\mathcal{D}(A_Y)$ to be the set of all functions $f \in C^0(\bar{\Omega}_B)$ such that $A_Y f \in C^0(\bar{\Omega}_B)$, f is constant on $\partial\Omega_B$ and

$$\alpha \int_{\partial\Omega_B} a_B(y) \nabla f(y) \cdot \nu(y) = A_Y f(y') \quad \text{for all } y' \in \partial\Omega_B \,.$$

The condition that f is constant on $\partial\Omega_B$ identifies the boundary to a point topologically, and hence the generated process Y makes random jumps on the boundary whenever it hits $\partial\Omega_B$.

The process Y is not a semi-martingale, because it has infinitely many of these $O(1)$ jumps in any time interval where it hits the boundary. However, if $g \in C^2(\bar{\Omega}_B)$ is any function that is *constant* on $\partial\Omega_B$, then $g(Y)$ *is* a continuous semi-martingale, and we can obtain an SDE for $g(Y)$. Let the local time Y on $\partial\Omega_B$, be denoted by ℓ, which satisfies

$$d\ell_t = \alpha \mathbf{1}_{\{Y_t \in \partial\Omega_B\}} \, dt \,. \tag{7.4}$$

Then, if $g \in C^2(\bar{\Omega}_B)$ is a function that is constant on $\partial\Omega_B$, then the process $g(Y)$ is a continuous semi-martingale and satisfies the SDE

$$dg(Y_t) = \mathbf{1}_{\{Y_t \in \Omega_B\}} \nabla g(Y_t) \cdot \sigma_B(Y_t) \, dW_t + \mathbf{1}_{\{Y_t \in \Omega_B\}} A_Y g(Y_t) \, dt \\ + \alpha \mathbf{1}_{\{Y_t \in \partial\Omega_B\}} \Big(\int_{\partial\Omega_B} a_B(y) \nabla g(y) \cdot d\nu(y) \Big) \, dt \,.$$

Let $\bar{W}$ be a Brownian motion independent of Y, and define

$$X_t \stackrel{\text{def}}{=} \bar{\sigma}_F \bar{W}_{\ell_t}$$

where $\bar{\sigma}$ is such that $\bar{a}_F = \frac{1}{2} \bar{\sigma}_F \bar{\sigma}_F^*$ where $\bar{a}_F$ is the constant homogenized matrix obtained from the corrector problems described in §1.1. The process X is not Markov, but the pair $Z \stackrel{\text{def}}{=} (X, Y)$ is.

Let

$$A_Z f = A_Z f(x, y) \stackrel{\text{def}}{=} \mathbf{1}_{\{y \in \partial\Omega_B\}} \Big(\nabla_x \cdot \bar{a}_F \nabla_x f + \alpha \int_{\partial\Omega_B} a_B(y') \nabla_y f \cdot d\nu(y') \Big) \\ + \mathbf{1}_{\{y \in \Omega_B\}} \nabla_y \cdot a_B(y) \nabla_y f$$

and define $\mathcal{D}(A_Z)$ to be the set of all functions $f \in C_0(\mathbb{R}^2 \times \bar{\Omega}_B) \cap C^2(\mathbb{R}^2 \times \Omega_B)$ such that $A_Z f \in C_0(\mathbb{R}^2 \times \Omega_B)$ and $f(x, \cdot)$ is constant on $\partial\Omega_B$. The process $Z \stackrel{\text{def}}{=} (X, Y)$ is the Fellerian Markov process with generator A_Z and domain $\mathcal{D}(A_Z)$.

8. Appendix

8.1. **The Two-Scale Homogenization of PDE.** Here we prove the convergence of the solution to the comb model PDE using the two-scale convergence method as in [Cla98]. To begin, let $\Omega \stackrel{\text{def}}{=} (a, b)$ where $a < b$ are real numbers, $\Omega'_\varepsilon = (\mathbb{R} \times \{0\}) \cup (\varepsilon\mathbb{Z} \times (0, \varepsilon))$ and define $\Omega_\varepsilon = \Omega'_\varepsilon \cap (\Omega \times \mathbb{R})$. Define $Y \stackrel{\text{def}}{=} (\mathbb{T} \times \{0\}) \cup (\{0\} \times (0, 1]) \stackrel{\text{def}}{=} Y_1 \cup Y_2$.

Definition 8.1. Let $u^\varepsilon\colon \Omega_\varepsilon \times (0,T) \to \mathbb{R}$ and $u\colon \Omega \times Y \times (0,T) \to \mathbb{R}$. We say (u^ε) *two-scale converges* to u if for all $\psi \in C_c^\infty\big(\Omega \times Y \times (0,T)\big)$ we have

$$\int_{\Omega_\varepsilon \times (0,T)} u^\varepsilon(x_1,x_2,t)\psi\Big(x, \frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, t\Big)\, d\mathcal{H}^1 \otimes dt \xrightarrow{\varepsilon\to 0} \int_{\Omega\times Y\times(0,T)} u(x,y,t)\psi(x,y,t)\, dx \otimes \mathcal{H}^1(dy) \otimes dt\,,$$

where $\mathcal{H}^1$ is the 1 dimensional Hausdorff measure. When this holds we write $u^\varepsilon \xrightarrow{2} u$.

Remark 8.2. Here we did not rescale the teeth of the comb Ω_ε because it would have made the two-scale definition and compactness results look unnatural.

We consider u^ε which is a solution to the Dirichlet problem related to the comb model given by

$$\partial_t u^\varepsilon - \frac{\varepsilon^2}{2}\partial_y^2 u^\varepsilon = 0, \quad (x,y,t) \in \varepsilon\mathbb{Z} \cap \Omega \times (0,\varepsilon) \times (0,T) \tag{8.1}$$

$$\partial_t u^\varepsilon - \frac{1}{2}\partial_x^2 u^\varepsilon = 0, \quad (x,y,t) \in \Omega \times \{0\} \times (0,T) \tag{8.2}$$

with flux conditions

$$\partial_y u(x,\varepsilon) = 0, \qquad (x,t) \in \varepsilon\mathbb{Z} \cap \Omega \times (0,T) \tag{8.3}$$

$$\frac{\varepsilon^2}{2}\partial_y u^\varepsilon + \frac{1}{2}\partial_x^+ u^\varepsilon - \frac{1}{2}\partial_x^- u^\varepsilon = 0, \quad (x,y,t) \in \varepsilon\mathbb{Z} \cap \Omega \times \{0\} \times (0,T) \tag{8.4}$$

and initial and boundary data

$$u^\varepsilon(x,y,0) = U_0\Big(x, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)$$
$$u^\varepsilon(a,0,t) = u^\varepsilon(b,0,t) = 0\,.$$

The following compactness results are the standard tools for the two-scale convergence method and a version of them can be found in [All92, Theorem 1.2, Proposition 1.14].

Theorem 8.3. *Let $u^\varepsilon : \Omega_\varepsilon \times (0,T) \to \mathbb{R}$.*

(i) If
$$\limsup_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} < \infty\,,$$
then up to a subsequence, there exists $u_0 \in L^2(\Omega\times Y)$ such that $u^\varepsilon \xrightarrow{2} u_0$.

(ii) If
$$\limsup_{\varepsilon\to 0} \quad \|u^\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} < \infty\,,$$
then up to a subsequence, there exists $u : \Omega \to \mathbb{R}$ independent of y and $u_1 \in L^2\big(0,T; L^2(\Omega) \times H^1(Y)/\mathbb{R}\big)$ such that $u^\varepsilon \xrightarrow{2} u(x)$ and $\nabla u^\varepsilon \xrightarrow{2} \nabla_x u(x) + \nabla_y u_1(x,y)$.

(iii) If
$$\limsup_{\varepsilon\to 0}\Big(\|u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))} + \varepsilon\|\nabla u^\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))}\Big) < \infty$$
then up to subsequence there exists $u_0 \in L^2((0,T)\times\Omega; H^1(Y))$ such that $u^\varepsilon \xrightarrow{2} u_0$ and $\varepsilon\nabla u^\varepsilon \xrightarrow{2} \nabla_y u_0$.

The standard parabolic apriori estimates for equations (8.1)-(8.4) yield

$$\sup_{\varepsilon}\|u^{\varepsilon}\|_{L^2(\Omega_\varepsilon\times(0,T))}<\infty$$
$$\sup_{\varepsilon}\|\partial_{x_1}u^{\varepsilon}\|_{L^2(\Omega\times\{0\}\times(0,T))}<\infty$$
$$\sup_{\varepsilon}\|\varepsilon\partial_{x_2}u^{\varepsilon}\|_{L^2((\Omega_\varepsilon\cap\{x_2>0\})\times(0,T)}<\infty.$$

So using the above compactness results we can find

$$v\in L^2\left((0,T)\times\Omega;H_0^1(Y_2)\right)\ \text{and}\ u_1\in L^2\left((0,T)\Omega;H^1(Y)/\mathbb{R}\right)$$

such that

$$u^{\varepsilon}\xrightarrow{2}u(x,t)+v(x,y,t)$$
$$\partial_{x_1}u^{\varepsilon}(x,0,t)\xrightarrow{2}\partial_x u(x,t)+\partial_{y_1}u_1(x,y_1,0,t)$$
$$\varepsilon\mathbf{1}_{\{x_2>0\}}\partial_{x_2}u^{\varepsilon}\xrightarrow{2}\partial_{y_2}v(x,y,t)$$

Consider test functions of the following forms

$$\psi_0(x,t)\in C^{\infty}\left(0,T;C_c^{\infty}\left(\Omega\right)\right)\ \text{with}\ \psi_0(x,T)=0$$
$$\psi_1(x,y,t)\in C^{\infty}\left(0,T;C_c^{\infty}\left(\Omega\times Y\right)\right)\ \text{with}\ \psi_1(x,y,T)=0$$
$$\psi_2(x,y,t)\in C^{\infty}\left(0,T;C_c^{\infty}\left(\Omega\times Y_2\right)\right)\ \text{with}\ \psi_2(x,y,T)=0.$$

Using the weak formulation of the comb model we use $\psi_0(x,t)$ as a test function and obtain

$$\begin{aligned}0=&-\int_0^T\int_{\Omega_\varepsilon}u^{\varepsilon}(x)\partial_t\psi_0(x_1,0)-\int_{\Omega_\varepsilon}U_0\Big(x_1,\frac{x}{\varepsilon}\Big)\psi_0(x_1,0)\\&+\frac{1}{2}\int_0^T\int_{\Omega_\varepsilon\cap\{x_2=0\}}\partial_{x_1}u^{\varepsilon}\partial_{x_1}\psi_0+\frac{\varepsilon^2}{2}\int_0^T\int_{\Omega_\varepsilon\cap\{x_2>0\}}\partial_{x_2}u^{\varepsilon}\partial_{x_2}\psi_0\\\to&-\int_0^T\int_{\Omega}\int_Y\left(u(x,t)+v(x,y,t)\right)\partial_t\psi_0(x,t)-\int_{\Omega}\int_Y U_0(x,y)\psi_0(x,0)\\&+\frac{1}{2}\int_0^T\int_{\Omega}\int_{Y_1}\left(\partial_x u(x,t)+\partial_{y_1}u_1(x,y,t)\right)\partial_x\psi_0(x,t)\,.\end{aligned}$$

Next we use $\varepsilon\psi_1(x,x/\varepsilon,t)$ as a test function which gives

$$\begin{aligned}0=&-\varepsilon\int_0^T\int_{\Omega_\varepsilon}u^{\varepsilon}\partial_t\psi_1-\varepsilon\int_{\Omega_\varepsilon}U_0\left(x_1,\frac{x}{\varepsilon}\right)\psi_1\left(x_1,\frac{x}{\varepsilon},0\right)\\&+\frac{\varepsilon}{2}\int_0^T\int_{\Omega_\varepsilon\cap\{x_2=0\}}\partial_{x_1}u^{\varepsilon}\left(\partial_{x_1}\psi_1+\frac{1}{\varepsilon}\partial_{y_1}\psi_1\right)+\frac{\varepsilon^2}{2}\int_0^T\int_{\Omega_\varepsilon\cap\{x_2>0\}}\partial_{x_2}u^{\varepsilon}\partial_{y_2}\psi_1\\&\to\frac{1}{2}\int_0^T\int_{\Omega}\int_{Y_1}\left(\partial_x u(x,t)+\partial_{y_1}u_1(x,y,t)\right)\partial_{y_1}\psi_1(x,y,t)\,.\end{aligned}$$

Finally from $\psi_2(x,x/\varepsilon,t)$ we see

$$\begin{aligned}0=&-\int_0^T\int_{\Omega_\varepsilon}u^{\varepsilon}\partial_t\psi_2-\int_{\Omega_\varepsilon}U_0\Big(x_1,\frac{x}{\varepsilon}\Big)\psi_2\Big(x_1,\frac{x}{\varepsilon},0\Big)\\&+\frac{\varepsilon^2}{2}\int_0^T\int_{\Omega_\varepsilon\cap\{x_2>0\}}\partial_{x_2}u^{\varepsilon}\frac{1}{\varepsilon}\partial_{y_2}\psi_1\end{aligned}$$

$$\to -\int_0^T \int_\Omega \int_Y (u(x,t)+v(x,y,t))\,\partial_t\psi_2(x,y,t) - \int_\Omega \int_Y U_0(x,y)\psi_2(x,y,0) + \frac{1}{2}\int_0^T \int_\Omega \int_{Y_2} \partial_{y_2} v(x,y,t)\partial_{y_2}\psi_2(x,y,t)\,.$$

This tells us that in the distributional sense we have

$$\int_Y \partial_t u(x,t) + \partial_t v(x,y,t)\,\mathcal{H}^1(dy) - \frac{1}{2}\int_{Y_1} \partial_x \left(\partial_x u(x,t) + \partial_{y_1} u_1(x,y,t)\right)\,\mathcal{H}^1(dy) = 0\,, \tag{8.5}$$

with

$$-\frac{1}{2}\partial_{y_1}^2 u_1(x,y,t) = 0 \qquad \text{for } y \in Y_1\ , \tag{8.6}$$

$$\partial_t u(x,t) + \partial_t v(x,y,t) - \frac{1}{2}\partial_{y_2}^2 v(x,y,t) = 0 \qquad \text{for } y \in Y_2\ , \tag{8.7}$$

$$u(x,0) + v(x,y,0) = U_0(x,y)\ , \tag{8.8}$$

$$\partial_{y_2} v(x,(0,1),t) = 0\ . \tag{8.9}$$

If we integrate (8.7) over Y_2, and use (8.9), we obtain

$$\int_{Y_2} (\partial_t u(x,t) + \partial_t v(x,y,t))\mathcal{H}^1(dy) + \frac{1}{2}\partial_{y_2} v(x,0,t) = 0.$$

We then substitute this into (8.5) we have

$$\partial_t u(x,t) - \frac{1}{2}\partial_{y_2} v(x,0,t) - \frac{1}{2}\int_{Y_1} \partial_x(\partial_x u(x,t) + \partial_{y_1} u_1(x,y,t)) = 0.$$

By the periodicity of u_1 in Y we see

$$\partial_t u(x,t) - \frac{1}{2}\partial_{y_2} v(x,0,t) - \frac{1}{2}\partial_x^2 u(x,t) = 0.$$

Combining all of this we find that for $U(x,y,t) = u(x,t) + v(x,(0,y),t)$ we have the equation stated earlier, namely

(8.10)
$$\begin{cases} \partial_t U - \frac{1}{2}\partial_y^2 U = 0 & (x,y,t) \in \mathbb{R}\times(0,1)\times(0,T) \\ U(x,y,0) = U_0(x,y) & (x,y) \in \mathbb{R}\times(0,1) \\ \partial_t U(x,0,t) - \frac{1}{2}\partial_x^2 U(x,0,t) - \frac{1}{2}\partial_y U(x,0,t) = 0 & (x,t) \in \mathbb{R}\times(0,T) \\ \partial_y U(x,1,t) = 0 & (x,t) \in \mathbb{R}\times(0,T) \end{cases}$$

8.2. **Discrete Comb Model.** Here we define a spatially discrete, continuous time random walk version on the comb model with exponential jump rates. Viewing this as a time changed random walk on $\varepsilon\mathbb{Z}$, we prove that the one dimensional distributions of the time change converges to the local time of doubly reflected sticky Brownian motion by direct computation of the Laplace transforms. For $\varepsilon > 0$ with $\frac{1}{\varepsilon} \in \mathbb{N}$ we consider a continuous time random walk $Z^\varepsilon(t) = (X^\varepsilon(t), Y^\varepsilon(t))$ on $\varepsilon\mathbb{Z} \times \{0, \varepsilon^2, 2\varepsilon^2, \ldots, k\varepsilon\}$ for some $k \in \mathbb{N}$. The transition rates are as follows, when $Y^\varepsilon(t) = 0$, we run exponential clocks τ_1, τ_2, τ_3 with rates $\frac{1}{2\varepsilon^2}, \frac{1}{2\varepsilon^2}, \frac{\alpha}{2\varepsilon}$ respectively when τ_1 rings we jump left, when τ_2 rings we jump right and when τ_3 rings we jump upwards. When $Y^\varepsilon(t) \notin \{0, k\varepsilon\}$ we run two exponential clocks with rates $\frac{1}{2\varepsilon^2}$ to decide when to jump up or down and when $Y^\varepsilon = k\varepsilon$ we run a clock with jump rate

$\frac{1}{\varepsilon^2}$ and jump down when it rings. We say Z^ε is "trapped" when $Y^\varepsilon > 0$.

The parameters are chosen to capture the behavior of the double porosity model (and hence the continuous comb model). The trap depth is $O(\varepsilon)$ and diffusion in the traps is $O(\varepsilon^2)$. We are taking further refinements in the trap with $\varepsilon \to 0$ to capture the diffusive behavior in the traps. The skew probability of entering the traps in the comb model is reflected in $\boldsymbol{E}(\tau_3) = O(\varepsilon)$.

8.3. **Taking the Limit.** Let $\{e_i^{\alpha/(2\varepsilon)}\}_{i=1}^\infty$, be independent exponential random variables with rates $\frac{\alpha}{2\varepsilon}$ and let $\{T_i\}_{i=1}^\infty$ be i.i.d. with common distribution given by

$$T = \sum_{i=1}^{N} e_i^{1/(2\varepsilon^2)}$$

where N is an independent copy of the exit time of a simple random walk from the interval $[0, \frac{2k}{\varepsilon}]$ starting at 1. Then T has the distribution of the exit time from one of the traps. Then let S_n^ε be the time for n visits to traps, i.e.

$$S_n^\varepsilon = T_1 + \cdots + T_n + e_1^{\alpha/(2\varepsilon)} + \cdots + e_n^{\alpha/(2\varepsilon)}$$

and we claim $S^\varepsilon_{\frac{x}{\varepsilon}}$ converges in distribution to U_x where U is a Levy process. We check this directly by proving convergence of the Laplace transforms. From [Fel57, pg.350], we see the Laplace transform of S_n^ε takes the form

$$\boldsymbol{E}(e^{-\lambda S_n^\varepsilon}) = \frac{1}{(1+\frac{2\varepsilon\lambda}{\alpha})^n}\left[\frac{\lambda_2(\lambda_1^{2k/\varepsilon}-1)+\lambda_1(1-\lambda_2^{2k/\varepsilon})}{\lambda_1^{2k/\varepsilon}-\lambda_2^{2k/\varepsilon}}\right]^n. \tag{8.11}$$

where

$$\lambda_1 = 1+2\varepsilon^2\lambda+\sqrt{(1+2\varepsilon^2\lambda)^2-1} = 1+2\sqrt{\lambda}\varepsilon+2\varepsilon^2\lambda+o(\varepsilon^2)$$
$$\lambda_2 = 1+2\varepsilon^2\lambda-\sqrt{(1+2\varepsilon^2\lambda)^2-1} = 1-2\sqrt{\lambda}\varepsilon+2\varepsilon^2\lambda+o(\varepsilon^2).$$

Next notice

$$\left(1+\frac{x}{n}\right)^n = e^{x-\frac{x^2}{2n}+o(\frac{1}{n})}$$

and so

$$\begin{aligned}\lambda_1^{2k/\varepsilon} &= \left(1+\frac{4k\sqrt{\lambda}+4k\lambda\varepsilon+o(\varepsilon)}{\frac{2k}{\varepsilon}}\right)^{\frac{2k}{\varepsilon}}\\ &= \exp\left(4k\sqrt{\lambda}+4k\lambda\varepsilon-\frac{\varepsilon}{4k}(4k\sqrt{\lambda}+4k\lambda\varepsilon)^2+o(\varepsilon)\right)\\ &= \exp(4k\sqrt{\lambda}+o(\varepsilon)) = \exp(4k\sqrt{\lambda})+o(\varepsilon).\end{aligned}$$

Similarly

$$\lambda_2^{2k/\varepsilon} = \exp(-4k\sqrt{\lambda})+o(\varepsilon).$$

Therefore,

$$\begin{aligned}\frac{\lambda_2(\lambda_1^{2k/\varepsilon}-1)+\lambda_1(1-\lambda_2^{2k/\varepsilon})}{\lambda_1^{2k/\varepsilon}-\lambda_2^{2k/\varepsilon}} &= \frac{(1-2\sqrt{\lambda}\varepsilon)(e^{4k\sqrt{\lambda}}-1)}{e^{4k\sqrt{\lambda}}-e^{-4k\sqrt{\lambda}}}\\ &\quad+\frac{(1+2\sqrt{\lambda}\varepsilon)(1-e^{-4k\sqrt{\lambda}})}{e^{4k\sqrt{\lambda}}-e^{-4k\sqrt{\lambda}}}+o(\varepsilon)\end{aligned}$$

$$= 1 + 2\sqrt{\lambda}\left(\frac{2 - e^{4k\sqrt{\lambda}} - e^{-4k\sqrt{\lambda}}}{e^{4k\sqrt{\lambda}} - e^{-4k\sqrt{\lambda}}}\right)\varepsilon + o(\varepsilon)$$

So we finally see

$$\begin{aligned}\boldsymbol{E}(e^{-\lambda S^{\varepsilon}_{\frac{x}{\varepsilon}}}) &= \frac{1}{(1+\frac{2\varepsilon\lambda}{\alpha})^{\frac{x}{\varepsilon}}}\left[\frac{\lambda_2(\lambda_1^{2k/\varepsilon}-1)+\lambda_1(1-\lambda_2^{2k/\varepsilon})}{\lambda_1^{2k/\varepsilon}-\lambda_2^{2k/\varepsilon}}\right]^{\frac{x}{\varepsilon}} \\ &\to \exp\left(-\left(\frac{2\lambda}{\alpha} + 2\sqrt{\lambda}\left(\frac{\cosh(4k\sqrt{\lambda})-1}{\sinh(4k\sqrt{\lambda})}\right)\right)x\right).\end{aligned}$$

Using the identity,

$$\frac{\cosh(2x)-1}{\sinh(2x)} = \frac{\sinh(x)}{\cosh(x)}$$

we have

$$\boldsymbol{E}(e^{-\lambda S^{\varepsilon}_{x/\varepsilon}}) \to \exp\left(-\left(\frac{2\lambda}{\alpha} + 2\sqrt{\lambda}\tanh(2k\sqrt{\lambda})\right)x\right) \text{ as } \varepsilon \to 0$$

Let $\hat{X}^{\varepsilon}(t)$ be a copy of the symmetric random walk on $\varepsilon\mathbb{Z}$ with jump rates $\frac{1}{2\varepsilon^2}$ and $R^{\varepsilon}(t) = \mathcal{L}^1(\{t : Y^{\varepsilon}(t) = 0\})$ so $X^{\varepsilon}(t)$ has the same distribution as $\hat{X}^{\varepsilon}(R^{\varepsilon}(t))$. Since we have path-wise convergence of $\hat{X}^{\varepsilon}$ to a Brownian motion, we need only find the distributional limit of $R^{\varepsilon}(t)$. To do this we first introduce $N^{\varepsilon}(t) = \sup\{k : S^{\varepsilon}_k \leqslant t\}$. Then we see

$$\mathbb{P}(\varepsilon N^{\varepsilon}(t) \geqslant x) = \mathbb{P}(S^{\varepsilon}_{x/\varepsilon} \leqslant t) \to \mathbb{P}(U_x \leqslant t) = \mathbb{P}(U_t^{-1} \geqslant x)$$

and so $\varepsilon N^{\varepsilon}(t) \to U_t^{-1}$ in distribution. We claim that $R^{\varepsilon}(t) = e_1^{\alpha/(2\varepsilon)} + \cdots + e_{N^{\varepsilon}(t)}^{\alpha/(2\varepsilon)} \to \frac{1}{\alpha}U_t^{-1}$ where the e_i's are related to $N^{\varepsilon}(t)$ as in the definitions of S^{ε} and $N^{\varepsilon}(t)$.

We want to apply the strong law of large numbers so take a sequence $\varepsilon_k \to 0$. Let $T_i^{\varepsilon_k}$ be independent for all i, k, $e_i^{\alpha/(2\varepsilon_k)} := \varepsilon_k e_i^{\alpha/2}$ and $S_n^{\varepsilon_k}, N^{\varepsilon_k}(t)$ defined as before using these random variables. Then as $N^{\varepsilon_k}(t) \to \infty$ almost surely as $k \to \infty$ we have

$$R^{\varepsilon_k}(t) = \varepsilon_k N^{\varepsilon_k}(t)\left(\frac{e_1^{\alpha/2} + \ldots e_{N^{\varepsilon_k}(t)}^{\alpha/2}}{N^{\varepsilon_k}(t)}\right) \to \frac{2}{\alpha}U_t^{-1}$$

in distribution.

We note that $\sqrt{2\lambda}\tanh(k\sqrt{2\lambda})$ is the Laplace exponent for the inverse local time at 0 of a doubly reflected Brownian motion on $[0,k]$. Therefore U is the sum of a drift and the inverse of a local time i.e. $U_t = \frac{t}{\alpha} + (L^B)_t^{-1}(0)$ where B is a doubly reflected BM on $[0,k]$.

8.4. **Two-Scale limit.** We now see that we have

$$\left(X^{\varepsilon}(t), \frac{1}{\varepsilon}Y^{\varepsilon}(t)\right) \to \left(W_{\frac{2}{\alpha}U_t^{-1}}, B_{t-\frac{2}{\alpha}U_t^{-1}}\right)$$

but we can simplify this by looking at $t - (2/\alpha)U_t^{-1}$ more carefully.

Proposition 8.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing continuous function and let f^{-1} denote the right continuous inverse of f. Then for all $t \in \mathbb{R}$*

$$t = (s + f(s))^{-1}(t) + \left(s + f^{-1}(s)\right)^{-1}(t) =: x_1 + x_2$$

Proof. By definition we have

$$x_1 + f(x_1) = t \tag{8.12}$$

$$x_2 = \inf\{x : x + f^{-1}(x) > t\} \tag{8.13}$$

First we note $f^{-1}(f(x_1)) > x_1$ and by (8.12) we have $f(x_1) = t - x_1$. Hence $t - x_1 + f^{-1}(t - x_1) > t$ and (8.13) implies $t - x_1 \geqslant x_2$. The definition of x_2 tells us $x_2 + f^{-1}(x_2) \geqslant t$ and hence by continuity of f, we know $x_2 \geqslant f(t - x_2)$. Adding $t - x_2$ to either side yields $t \geqslant t - x_2 + f(t - x_2)$ and so $x_1 \geqslant t - x_2$ by (8.12). □

This proposition shows us that we can write

$$t - \frac{2}{\alpha} U_t^{-1} = t - \left(s + (L^B)^{-1}_{\alpha s/2}(0)\right)^{-1}(t) = \left(s + \frac{2}{\alpha} L^B_s(0)\right)^{-1}(t)$$

Brownian motion with this time change is a well studied process known as a Sticky Brownian motion. Here we have a doubly reflected Brownian motion which is sticky at 0.

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