Assignment 6 (assigned 2019-02-21, due 2019-02-28).

1. Let S be a geometric Brownian motion with mean return rate α and volatility σ. Let $\gamma > 0$ and consider a security that pays $S(T)^\gamma$ at time T. Compute the arbitrage free price of this security.

   HINT: Use the replicating portfolio argument to reduce this problem to finding the solution of a PDE with appropriate terminal and boundary conditions. Now look for a solution to these equations that is of the form $c(t,x) = f(t)g(x)$ for some functions f, g, and then find f and g explicitly.

2. *Question asked on a job interview (a few years ago)*

   > Determine the final value of a delta-hedge of a long call position if the realized volatility is different from the implied volatility.

   The question asked was the sentence above. Here is the same question posed in more detail. Let

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$$

be the price of a European call, expiring at time T with strike price K, if the stock price at time t is x, where

$$d_\pm(T-t,x) = \frac{1}{\sigma_1\sqrt{T-t}}\Big[\log\frac{x}{K} + \Big(r \pm \frac{1}{2}\sigma_1^2\Big)(T-t)\Big].$$

This call price formula assumes the underlying stock is a geometric Brownian motion with volatility $\sigma_1 > 0$. For this problem we take this to be the market price of the call. In other words, σ_1 is the *implied volatility*, the one that makes the Black-Scholes formula agree with the market price of the call.

Suppose, however, that the underlying stock is really a geometric Brownian motion with volatility $\sigma_2 > 0$, i.e.,

$$dS(t) = \alpha S(t)\,dt + \sigma_2 S(t)\,dW(t).$$

We assume for most of this problem that σ_2 is constant. After we observe the stock price between times 0 and T, if we estimate the so-called *realized volatility*, we get σ_2. Consequently, the market price of the call at time zero is incorrect, although we do not know this at time zero.

We set up a portfolio whose value at each time t we denote by $X(t)$. We begin with $X(0) = 0$. At each time t, the portfolio is long one European call and is short $\partial_x c(t, S(t)) = N(d_+(T-t, S(t)))$ shares of stock. This is the delta-hedge of the long call position.

There is a cash position associated with this hedge which is often neglected. Here we keep track of it. We start with zero initial capital, and so at the initial time the portfolio has a cash position

$$-c(0,S(0)) + S(0)\partial_x c(0,S(0)) = Ke^{-rT}N\big(d_-(T,S(0))\big),$$

because we spend $c(0,S(0)) = S(0)N(d_+(T,S(0))) - Ke^{-rT}N(d_-(T,S(0)))$ to buy the call and we receive $S(0)\partial_x c(0,S(0)) = S(0)N(d_+(T,S(0)))$ when we short the stock. This cash is invested in a money market account with a constant continuously compounding interest rate r. At subsequent times, as we adjust the position in stock, we finance this by taking money from the money market account or depositing money into the money market account, depending on whether we are buying or selling stock, respectively. Therefore, the differential of the portfolio value is

$$dX(t) = dc(t,S(t)) - \partial_x c(t,S(t))\,dS(t) + r\big[X(t) - c(t,S(t)) + S(t)\partial_x c(t,S(t))\big]\,dt$$

for $0 \leqslant t \leqslant T$. The term $dc(t,S(t))$ accounts for the profit or loss from the long call position. The term $-\partial_x c(t,S(t))\,dS(t)$ accounts for the profit or loss from the short stock position. Finally, since $X(t)$ is the total portfolio value, if we take into account the long call and the short stock positions, we see that the cash position is

$$X(t) - c(t,S(t)) + S(t)\partial_x c(t,S(t)).$$

This is invested at interest rate r. The term

$$r\Big[X(t) - c(t,S(t)) + S(t)\partial_x c(t,S(t))\Big]\,dt$$

in the above formula for $dX(t)$ keeps track of these interest earnings.

   (a) Determine the value of $X(T)$. In particular, discuss the relationship among σ_1, σ_2 and the sign of $X(T)$. HINT: Compute $d(e^{-rt}X(t))$.

   (b) How would the analysis change if, instead of being constant, σ_2 is an adapted process $\sigma_2(t)$?

3. *(Asian options)* Let S be a geometric Brownian motion with mean return rate α and volatility σ, modelling the price of a stock. Let $Y(t) = \int_0^t S(s)\,ds$.

   (a) Let $f = f(t,x,y)$ be any function that is C^2 in x,y and C^1 in t. Find a condition on f such that $X(t) = f(t, S(t), Y(t))$ represents the wealth of an investor that has a portion of his wealth invested in the stock, and the rest in a money market account with return rate r.

   HINT: We know that if X represents the wealth such an investor and $\Delta(t)$ is the number of shares of the stock held at time t, then $dX(t) = \Delta(t)\,dS(t) + r(X(t) - \Delta(t)S(t))\,dt$.

   Let $V = V(x,y)$ be a function and consider a derivative security that pays $V(S(T), Y(T))$ at time T. Note, if $V(x,y) = (y/T - K)^+$ then this is exactly an Asian option with strike price K.

   (b) Suppose $c = c(t,x,y)$ is a function such that $c(t,S(t),Y(t))$ is the arbitrage free price of this security at time t. Assuming c is C^1 in t and C^2 in x,y when $t < T$, find a PDE and boundary conditions satisfied by c.

   [The PDE you obtain will be similar to the Black-Scholes PDE, but will also involve derivatives with respect to the new variable y. Unlike the case of European options, the PDE you obtain here will not have an explicit solution.]

   (c) Conversely, if c is the solution to the PDE you found in the previous part then show that the arbitrage free price of this security is exactly $c(t, S(t), Y(t))$.

4. Let W and B be two independent (one dimensional) Brownian motions. Let M, N be defined by

$$M(t) = \int_0^t W(s)\, dB(s) \qquad \text{and} \qquad N(t) = \int_0^t B(s)\, dW(s)\,.$$

Show $[M, N] = 0$. Also verify $\boldsymbol{E} M(t)^2 \boldsymbol{E} N(t)^2 \neq \boldsymbol{E} M(t)^2 N(t)^2$, and show that M, N are not independent even though $[M, N] = 0$.

5. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants. Let $K, T > 0$ and consider a derivative security that pays $(S(T)^2 - K)^+$ at maturity T. Compute the arbitrage free price of this security at any time $t \in [0, T)$. Your answer may involve r, σ, K, t, T, S, and the CDF of the normal distribution, but not any integrals or expectations.

   HINT: The simplest way to solve this problem is to use the risk neutral pricing formula, along with the explicit Black-Scholes formula you already know.