46-944 Stochastic Calculus for Finance I: Final.

2019-03-06

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 8 questions and 80 points.*
- *The questions are roughly ordered by difficulty.*
- *Please don't panic if you're running out of time, and do as much as you can correctly. It is possible to get a good grade, even if you don't finish the entire exam. Good luck* ☺.

Unless otherwise stated, W denotes a standard (one dimensional) Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ (if not otherwise specified) is the Brownian filtration.

[10] 1. Let $X(t) = e^{-tW(t)^2}$. Find function f such that $X(t) - \int_0^t f(s, W(s))\, ds$ is a martingale.

[10] 2. Let W and B be two independent standard one dimensional Brownian motions, and define the process X by

$$X(t) = W(t)^2 + W(t)B(t) + t^2\,.$$

Compute $[X, X](t)$ and $[X, W](t)$. Express each of your answers in the form $\int_0^t f(s, W(s), B(s))\, ds$, where you explicitly find a formula f.

[10] 3. Let $0 \leqslant s < t$. Compute $\boldsymbol{E}\big(W(s)\,|W(t)|\big)$. You may leave your answer as an unsimplified (non-random) Riemann integral involving s, t provided it does not involve W.

[10] 4. Consider a financial market consisting of a stock and a money market account. We assume the stock price, denoted by S, follows a geometric Brownian motion with mean return rate α and volatility σ for some given $\alpha \in \mathbb{R}$ and $\sigma \neq 0$. We know that until time $T_1 > 0$, the money market account will have (continuously compounded) interest rate r_1. After time T_1, the interest rate will become r_2. Here $r_1, r_2 \in \mathbb{R}$ are constants.

(a) Let $T > T_1$ and consider a European call on the stock with strike price K and maturity T. Write down the price of this option at time $t < T$ using the risk neutral pricing formula. You need not simplify / evaluate this formula. You should, however, express the risk neutral measure $\tilde{\boldsymbol{P}}$ in the form $d\tilde{\boldsymbol{P}} = Z(T)\, d\boldsymbol{P}$, and find a formula for that expresses $Z(T)$ explicitly in terms of W, T_1, T, r_1, r_2, α and σ without involving expectations or integrals.

(b) If $0 \leqslant t < T_1$, compute $\tilde{\boldsymbol{E}}(S(T) \mid \mathcal{F}_t)$. Here $\tilde{\boldsymbol{E}}$ denotes the conditional expectation with respect to the risk neutral measure $\tilde{\boldsymbol{P}}$. Express your answer in terms of r_1, r_2, T_1, T, t and S without involving expectations or integrals.

[10] 5. Find a continuous martingale M with $M(0) = 1$ such that the process X, defined by

$$X(t) = \big(W(t) + t^2\big)M(t)\,,$$

is also a martingale.

[10] 6. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate r, and the stock price, denoted by S, follows a geometric Brownian motion with mean return rate α and volatility σ. Here α, σ and $r > 0$ are constants. Let $K, T > 0$ and consider a security that matures at time T and pays \$1 if $S(T) \geqslant K$, and \$0 otherwise. Given $t \in [0, T]$, compute the arbitrage free price of this security at time t. Also compute $\Delta(t)$, the number of shares of the stock held in the replicating portfolio at time t. Express your answers in terms of α, σ, r, t, T, $S(t)$ and the CDF of the standard normal, without using expectations or integrals.

[10] 7. A simplified version of the Vasiček and Ho-Lee model stipulates that the interest rate $R(t)$ is given by

$$R(t) = r_0 + \theta t + \kappa \tilde{B}(t)\,,$$

where r_0, $\kappa > 0$, $\theta \in \mathbb{R}$ and $\tilde{B}$ is a Brownian motion under the risk neutral measure $\tilde{\boldsymbol{P}}$. Consider a bond that pays \$1 at maturity time T. Compute the arbitrage free price of this bond at time 0. Express your answer in terms of r_0, T, θ and κ, without involving expectations or integrals.

10 8. Let S be a geometric Brownian motion with mean return rate α and volatility σ. Given $T > 0$ and a non-random function f, the Markov property guarantees that there exists a non-random function g such that for any $t \leqslant T$ we have

$$\boldsymbol{E}\big(f(S(T)) \,\big|\, \mathcal{F}_t\big) = g(t, S(t))\,.$$

Find non-random functions h_1, h_2, h_3 (that may depend on x, t, α, and σ, but not on S, T, f or g) such that

$$\partial_t g(t,x) = h_1(t,x)\partial_x g(t,x) + h_2(t,x)\partial_x^2 g(t,x) + h_3(t,x)$$

Note: You are not required to find a formula for g itself.