# 46-944 Stochastic Calculus for Finance I: Final.

2017-12-16, Pittsburgh.

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 8 questions and 40 points.*
- *The questions are roughly ordered by difficulty.*
- *Please don't panic if you're running out of time, and do as much as you can correctly. It is possible to get a good grade, even if you don't finish the entire exam. Good luck* ◡̈.

Unless otherwise stated, $W$ denotes a standard (one dimensional) Brownian motion, and the filtration $\{\mathcal{F}_t \mid t \geqslant 0\}$ (if not otherwise specified) is the Brownian filtration.

[5] 1. Let $X(t) = W(t)^2$. Compute $[X, W](t)$. Express your answer in the form $\int_0^t f(s, W(s))\,ds$ for a function $f$ that you explicitly find the formula for.

[5] 2. Let $0 < s < t$. Compute $\boldsymbol{E}\big((W(s) + W(t))^2 \bigm| \mathcal{F}_s\big)$. Express your answer as a function of $s, t$ and $W$, without involving integrals or expected values.

[5] 3. Find all $\alpha \in \mathbb{R}$ for which the process $e^{\alpha t} \sin(5W(t))$ is a martingale.

[5] 4. Let $W = (W_1, W_2)$ be a standard two dimensional Brownian motion, and define

$$X(t) = t + \int_0^t \mathbf{1}_{\{W_1(s) > W_2(s)\}}\, dW_1 + \int_0^t \mathbf{1}_{\{W_1(s) \leqslant W_2(s)\}}\, dW_2 \,.$$

Compute $[X, X](t)$, and $\boldsymbol{E} \exp(7X(t))$. Express your answer as a function of $t$ without involving integrals or expected values.

[5] 5. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate $r$, and the stock price, denoted by $S$, follows a geometric Brownian motion with mean return rate $\alpha$ and volatility $\sigma$. Here $\alpha$, $\sigma$ and $r > 0$ are constants. Let $K_1, K_2, T > 0$ be constants with $K_1 < K_2$, and define

$$V(T) = \min\{(S(T) - K_1)^+, K_2\}\,.$$

Consider a derivative security that pays $V(T)$ at maturity time $T$. Compute the arbitrage free price of this security at any time $t \in [0, T)$.

[Express your answer in terms of the constants in the problem, $S$, standard functions (like logarithms, exponentials, etc.), and the CDF of the normal distribution. In particular, your answer should **not** involve any integrals, probabilities or expectations. If the answer is long, it is OK to write it in the form $f_1(t, S(t)) + f_2(t, S(t)) + \cdots$, provided you write down explicit formulae (as described above) for the each of the functions you use. **Note:** While you are free to use any method you prefer, there is a short and simple way to solve this problem.]

[5] 6. Compute $\boldsymbol{E}\Big[\Big(\int_0^t s\,dW(s)\Big)\Big(\int_0^t W(s)\,ds\Big)\Big]$. Express your answer as a function of $t$ without involving integrals or expected values.

[5] 7. Consider a financial market consisting of a stock and a money market account. Suppose the money market account has a constant return rate $r$, and the stock price, denoted by $S$, follows a geometric Brownian motion with mean return rate $\alpha$ and volatility $\sigma$. Here $\alpha$, $\sigma$ and $r > 0$ are constants. Let $T > 0$ and define

$$V(T) = \exp\Big(\frac{1}{T}\int_0^T \ln S(s)\,ds\Big)\,.$$

Consider a derivative security that pays $V(T)$ at maturity time $T$. Compute the arbitrage free price of this security at any time $t \in [0, T)$. Express your final answer *without* using conditional expectations. (It may, however, have expectations and integrals.)

[5] 8. Let $\alpha \geqslant 2$. Compute $\displaystyle\lim_{t \to s^+}\Big[\frac{1}{t-s}\Big(\boldsymbol{E}(W(t)^\alpha \mid \mathcal{F}_s) - W(s)^\alpha\Big)\Big]$. Your answer should not involve any limits, expectations or integrals.