Ex 0 (Fisk- Stratonovich)

$$dX_t = b_t^{(1)} dt + \sigma_t^{(1)} dW_t$$

$$dY_t = b_t^{(2)} dt + \sigma_t^{(2)} dW_t$$

Define the Fisk- Stratonovich integral of Y w.r.t X by

$$\int_0^t X_s \circ dY_t = \int_0^t Y_s \, dX_s + \frac{1}{2} \underbrace{\int_0^t \sigma_s^{(1)} \sigma_s^{(2)} ds}.$$

$f \in C^3$, show

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s$$

<u>Ex 2</u>: $W = (W^1, \dots, W^d)$ d-dim B.M.

Bessel process — $X_t = |W_t| = \sqrt{(W_t^1)^2 + \cdots + (W_t^d)^2}$

Show $dX_t = \frac{d-1}{2X_t}dt + dB_t$

where B is standard B.M.

$$f(x) = |x| = \sqrt{x_1^2 + \cdots + x_d^2}$$

$$\frac{\partial f}{\partial x_i}(x) = \frac{x_i}{|x|}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_i}(x) = \frac{|x| - \frac{x_i}{|x|}}{|x|^2} = \frac{|x|^2 - x_i^2}{|x|^3}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = -\frac{x_i x_j}{|x|^3}$$

$$dX_t = d(f(W_t)) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(W_t)\, dW_t^i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(W_t)\, d[W^i, W^j]_t$$

$$= \sum_{i=1}^{d} \frac{W_t^i}{|W_t|} dW_t^i + \frac{1}{2} \sum_{i=1}^{d} \frac{|W_t|^2 - (W_t^i)^2}{|W_t|^3} dt$$

$$= \frac{1}{2} \underbrace{\frac{(d-1)\,|W_t|^2}{|W_t|^3}\,dt}_{\frac{(d-1)}{2X_t}\,dt} + \underbrace{\sum_{i=1}^{d} \frac{W_t^i}{|W_t|}\,dW_t^i}_{B_t \text{ m'gale } \checkmark}$$

$$d[B,B]_t = \sum_{i=1}^{d} \frac{(W_t^i)^2}{|W_t|^2}\,dt = dt$$

$$= e^{r\tau} \int_{-\infty}^{L(y,\tau)} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2} \frac{1}{\sqrt{2\pi}}\, dz$$

$$+ \, y\left(1 - N(L(y,\tau))\right)$$

$$1 - N(L) = N(-L)$$

$$= e^{r\tau} N\left(L(y,\tau) - \sigma\sqrt{\tau}\right) + y\, N(-L(y,\tau))$$

(a) Under Q, $S_T = S_t \, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^Q - W_t^Q)}$

$$\mathbb{E}^Q\left(\min\left\{y, \frac{S_T}{S_t}\right\}\right) = \int_{-\infty}^{\infty} \min\left\{y, e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma \underbrace{(W_T^Q - W_t^Q)}_{\sqrt{T-t}\, Z}}\right\} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$y \geq e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z}$$

$$\Longleftrightarrow \boxed{\frac{\log(y) - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}} = L(y, T-t) \geq Z$$

$$= \int_{-\infty}^{L(y,T-t)} e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$+ \int_{L(y,T-t)}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

__Ex3__: BS-Model, $R_t \equiv r$. $T > 0$.

$S \sim GBM(\mu, \sigma^2)$.

$\mathbb{Q}$ r.n.m.

(a) Fix $y > 0$, $t \leq T$. Show

$$\boxed{\mathbb{E}^{\mathbb{Q}}\left(\min\left\{y, \frac{S_T}{S_t}\right\}\right) = e^{r\tau} N\left(L(y,\tau) - \sigma\sqrt{\tau}\right) + y\, N\left(-L(y,\tau)\right)}$$

$\tau = T - t$, N - cdf of $N(0,1)$.

$$L(y,\tau) = \frac{1}{\sigma\sqrt{\tau}}\left(\log(y) - \left(r - \frac{1}{2}\sigma^2\right)\tau\right)$$

(b) minimum option pays $\qquad t_0 \in (0, T)$

$$V_T = \min\{S_T, S_{t_0}\}$$

Find V_t for $t \leq t_0$

$$V_t = \mathbb{E}^{Q}\left(e^{-r(T-t)} V_T \mid \mathcal{F}_t\right) \qquad S_t = x \qquad V(t,x)$$

$$= \mathbb{E}^{Q}\left(e^{-r(T-t)} \min\{S_T, S_{t_0}\} \mid \mathcal{F}_t\right)$$

$$= \mathbb{E}^{Q}\left(\underline{e^{-r(T-t)} S_{t_0}} \min\left\{\frac{S_T}{S_{t_0}}, 1\right\} \mid \mathcal{F}_t\right)$$

$$\mathbb{E}^{Q}\left(\mathbb{E}^{Q}\left(\cdots\cdots \mid \mathcal{F}_{t_0}\right) \mid \mathcal{F}_t\right)$$

$$\frac{S_T}{S_{t_0}} = e^{(r-\frac{1}{2}\sigma^2)(T-t_0) + \sigma(W_T^{\mathbb{Q}} - W_{t_0}^{\mathbb{Q}})}$$

$$= \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)} S_{t_0} \, \mathbb{E}^{\mathbb{Q}}\left(\min\left\{ \frac{S_T}{S_{t_0}}, 1 \right\} \Big| \mathcal{F}_{t_0} \right) \Big| \mathcal{F}_t \right)$$

$$(\tau = T - t_0) = \mathbb{E}^{\mathbb{Q}}\left(e^{-r(T-t)} S_{t_0} \left\{ e^{r\tau} N(L(1,\tau) - \sigma\sqrt{\tau}) + N(-L(1,\tau)) \right\} \Big| \mathcal{F}_t \right)$$

$$e^{-rt_0}$$

$$= e^{-r(T-t)} e^{rt_0} e^{-rt} S_t \left\{ \cdots\cdots \right\}$$

$$= S_t \left(N(L(1,\tau) - \sigma\sqrt{\tau}) - e^{-r\tau} N(-L(1,\tau)) \right)$$

$$\tau = T - t_0$$

$$\mathbb{E}^{\mathbb{Q}}\left(|S_T - S_{t_0}| \,\big|\, \mathcal{F}_t \right)$$

$$\mathbb{E}^{\mathbb{Q}}\left(\min\{S_T, S_{t_0}\} \,\big|\, \mathcal{F}_t \right)$$

Ex4 :

$$c(t, x; K) = x N(d_+(T-t, x; K)) - K e^{-r(T-t)} N(d_-(T-t, x; K))$$

$$d_\pm(\tau, x; K) = \frac{1}{\sigma\sqrt{\tau}}\left(\log\left(\frac{x}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right)$$

Now, fix $0 < K < M$, look at "up & out" call option :

$$V_T = (S_T - K)^+ \mathbb{1}_{\{S_T \leq M\}}$$

Find $V(t,x)$ = Price of "up & out" call option at t, given $S_t = x$.

$f(x) = (x-K)^+ \mathbb{1}_{\{x<M\}}$

$$(x-K)^+ = (x-K)\mathbb{1}_{\{x \geq K\}}$$

$$(x-K)^+\mathbb{1}_{\{x<M\}} = (x-K)\mathbb{1}_{\{K \leq x < M\}}$$

$$= (x-K)\left(\mathbb{1}_{\{K \leq x\}} - \mathbb{1}_{\{M \leq x\}}\right)$$

$$\left(+ M\mathbb{1}_{\{M \leq x\}} - M\mathbb{1}_{\{M \leq x\}}\right)$$

$$= \underline{(x-K)^+} - (x-\underline{M)\mathbb{1}_{\{M \leq x\}}}$$

$$+ \underline{(K-M)\mathbb{1}_{\{M \leq x\}}}.$$

$$= (x-K)^+ - (x-M)^+ + (K-M)\mathbb{1}_{\{M \leq x\}}$$

Ex2: $V_T = S_T^p$

BS Model, $S \sim GBM(\mu, \sigma^2)$, T, $p > 1$

(a) find $\tilde{\mu}, \tilde{\sigma}$ s.t. $S^p \sim GBM(\tilde{\mu}, \tilde{\sigma}^2)$ under r.n.m Q

(b) find V_t

(c) initial capital x, Δ s.t.

$$X_t^{x,\Delta} = V_t$$

$$(*) \leq \boxed{e^{\int_0^T p\mu\hat{\pi}_u - \frac{1}{2}p(1-p)\sigma^2\hat{\pi}_u \, du}}$$

$$\mathbb{E}\left(\left(X_T^{\pi}\right)^p\right) = X_0^p \, \mathbb{E}\left(\underline{(**)}\,(*)\right)$$

$$\leq X_0^p \, \underbrace{\mathbb{E}\left((**)\right)}_{=1}$$

$$? \quad = \quad \leq \underline{X_0^p \, e^{\frac{p\mu^2}{2(1-p)\sigma^2}T}}$$

$$() = ()()$$

$$\frac{dS}{S} = \mu_t dt + \sigma_t dW_t$$

$$[M_1, M_2]_t = \int_0^t \Delta_1(s)\,\Delta_2(s)\, d[W_1, W_2]_s = 0$$

$$S_T = x e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma(W_T^Q - W_t^Q)}$$

$$V(t,x) = \mathbb{E}^Q\left(e^{-r(T-t)} V_T \,\middle|\, \mathcal{F}_t, S_t = x \right)$$

$$= \mathbb{E}^Q\left(e^{-r(T-t)} \underbrace{(S_T-K)^+ \mathbb{1}_{\{S_T < M\}}} \,\middle|\, \mathcal{F}_t, S_t = x \right)$$

$$= \mathbb{E}^Q\left(e^{-r(T-t)} \left((S_T - K)^+ - (S_T - M)^+ + (K-M)\mathbb{1}_{\{M \le S_T\}} \right) \,\middle|\, \mathcal{F}_t \right)_{S_t = x}$$

$$= C(t,x;K) - C(t,x;M) + (K-M)\, Q(M \le S_T)\, e^{-r(T-t)}$$

$$Q\left(M \le x e^{(r-\frac{1}{2}\sigma^2)(T-t) + \sigma\sqrt{T-t}\, Z} \right)$$

$$= Q\left(\frac{\log\left(\frac{M}{x}\right) - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \le Z \right)$$

$$-\left(\frac{\log\left(\frac{x}{M}\right) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$= -d_-(T-t, x; M)$$

$$Q(\cdots) = Q(-d_-(T-t, x; M) \leq Z)$$

$$= 1 - N(-d_-(T-t, x; M))$$

$$= N(d_-(T-t, x; M))$$

$$V(t,x) = \underline{c(t,x;K)} - c(t,x;M) + e^{-r(T-t)}(K-M)\, N(d_-(T-t; x, M))$$

Ex 5: (Utility maximization)

BS Model, $r = 0$

π_t = proportion of wealth invested in S

$X_0 > 0$ - initial capital.

(a) find dX_t^π.

$$X_t^\pi = \pi_t X_t^\pi + (1-\pi_t) X_t^\pi$$

$$dX_t^\pi = \frac{\pi_t X_t^\pi}{S_t} dS_t + \underbrace{\frac{(1-\pi_t) X_t^\pi}{B_t} dB_t}_{\equiv 0}$$

$$= \pi_t X_t^\pi (\mu dt + \sigma dW_t)$$

(b) $\max_{\pi} \mathbb{E}\left(U(X_T^{\pi})\right)$.

$U(x) = \frac{x^p}{p}$ $0 < p < 1$

utility function.

- $(X_T^{\pi})^p$

$X_t^{\pi} \sim$ generalized GBM $(\mu\pi_t, \sigma\pi_t)$

$$X_t^{\pi} = X_0 \, e^{\int_0^t (\mu\pi_u - \frac{1}{2}\sigma^2\pi_u^2)du + \int_0^t \sigma\pi_u dW_u}.$$

$$\underline{(X_T^{\pi})^p} = X_0^p \, e^{p\int_0^T \underline{\mu\pi_u - \frac{1}{2}\sigma^2\pi_u^2} du + \underline{p\int_0^T \sigma\pi_u dW_u}}$$

$$e^{\int_0^t \mu_s dW_s - \frac{1}{2}\int_0^t \mu_s^2 ds}$$

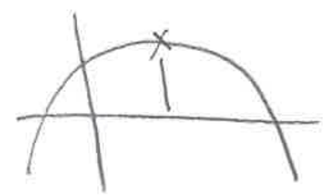
$$(X_T^\pi)^p = X_0^p \Big(\underbrace{e^{\int_0^T p\sigma\pi_u dW_u - \frac{1}{2}\int_0^T p^2\sigma^2\pi_u^2 du}}_{(**)}$$

$$\times \boxed{e^{\int_0^T \underbrace{\frac{1}{2}p^2\sigma^2\pi_u^2 + \left(\mu\pi_u - \frac{1}{2}\sigma^2\pi_u^2\right)p}_{(*)} du}}$$

$$\boxed{\underbrace{-\frac{1}{2}p(1-p)\sigma^2\pi_u^2 + p\mu\pi_u}}$$

$$\boxed{\hat{\pi}_u = \frac{p\mu}{\sigma^2 p(1-p)}}$$

$f \geq \sigma > 0$



$W =$

$M_t = \int_0^t f(W_s)\, dW_s$