STOCHASTIC CALCULUS

Derivative security that pays $h(S_T)$ at time T, the price of this derivative security at time t

$$\tilde{E}\left(e^{-r(T-t)} h(S_T) \mid \mathcal{F}_t\right) = v(t, S_t)$$

$$dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$$

Prove:

$$X_t = \sum_{i=1}^{3} \int_0^t \frac{W_s^i\,dW_s^i}{\sqrt{(W_s^1)^2 + (W_s^2)^2 + (W_s^3)^2}}$$ is a Brownian Motion.

$W^1 \perp\!\!\!\perp W^2 \perp\!\!\!\perp W^3$

$$X_t = \sum_{i=1}^{3} \int_0^t \frac{W_s^i}{\sqrt{(W_s^1)^2 + (W_s^2)^2 + (W_s^3)^2}}\,dW_s^i$$

$$X_t = X_t^1 + X_t^2 + X_t^3$$

$$X_t^i = \int_0^t \frac{W_s^i}{\sqrt{(W_s^1)^2 + (W_s^2)^2 + (W_s^3)^2}}\,dW_s^i$$

$1 \le i \le 3$

clearly X_t is a Martingale

$$\langle X \rangle_t = \langle X^1 \rangle_t + \langle X^2 \rangle_t + \langle X^3 \rangle_t$$

$$\left\langle \int_0^t Y_s\,dW_s^1, \int_0^t Z_s\,dW_s^2 \right\rangle$$

$$= \int_0^t Y_s Z_s\,d\langle W^1, W^2 \rangle_s$$

$$\left\langle \int_0^t Y_s\,dW_s, M_t \right\rangle = \int_0^t Y_s\,d\langle W, M \rangle_s$$

$$X_t = X_t^1 + X_t^2 + X_t^3$$

$$\langle X \rangle_t = \langle X^1 \rangle_t + \langle X^2 \rangle_t + \langle X^3 \rangle_t + 2\langle X^1, X^3 \rangle_t + 2\langle X^1, X^2 \rangle_t + 2\langle X^2, X^3 \rangle_t$$

$$\sqrt{(W_s^1)^2 + (W_s^2)^2 + (W_s^3)^2} = R_s$$

$$X_t^1 = \int_0^t \frac{W_s^1}{R_s}\, dW_s^1$$

$$X_t^2 = \int_0^t \frac{W_s^2}{R_s}\, dW_s^2$$

$$\langle X^1, X^2 \rangle = \int_0^t \frac{W_s^1 W_s^2}{R_s^2}\, d\langle W^1, W^2 \rangle_s$$

But W^1, W^2 & W^3 are independent

$$\Rightarrow \langle W^1, W^2 \rangle_s = 0,\ \langle W^1, W^3 \rangle_s = 0 \ \&\ \langle W^2, W^3 \rangle_s = 0$$

$$\langle X_t \rangle = \langle X^1 \rangle_t + \langle X^2 \rangle_t + \langle X^3 \rangle_t$$

$$\langle X^1 \rangle_t = \int_0^t \frac{(W_s^1)^2}{R_s^2}\, ds$$

$$\langle X^2 \rangle_t = \int_0^t \frac{(W_s^2)^2}{R_s^2}\, ds$$

$$\langle X^3 \rangle_t = \int_0^t \frac{(W_s^3)^2}{R_s^2}\, ds$$

$$\langle X \rangle_t = \int_0^t \frac{(W_s^1)^2 + (W_s^2)^2 + (W_s^3)^2}{R_s^2}\, ds \quad \text{But } R_s^2 = (W_s^1)^2 + (W_s^2)^2 + (W_s^3)^2$$

$$\langle X\rangle_t = \int_0^t ds = t$$

So X_t is a Martingale whose quadratic variation is t

Levy $\Rightarrow$ X_t is B·M

4.3) $V(T) = \frac{1}{T}\int_0^T S(t)\,dt$

$$\tilde{E}\left(e^{-r(T-t)}\frac{1}{T}\int_0^T S(u)\,du \;\middle|\; \mathcal{F}_t\right)$$

$$dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$$

Under the Risk Neutral Measure, ~~any~~ the drift of any traded security (stocks, options etc) that don't leak cash (dividends, coupons etc) have a drift of r. In other words, discounted traded securities are Martingales under the Risk Neutral Measure.

$$\tilde{E}\left(e^{-r(T-t)}\frac{1}{T}\left\{\int_0^t S(u)\,du + \int_t^T S(u)\,du\right\} \;\middle|\; \mathcal{F}_t\right)$$

$$\frac{e^{-r(T-t)}}{T}\left\{\int_0^t S(u)\,du + \tilde{E}\left(\int_t^T S(u)\,du \;\middle|\; \mathcal{F}_t\right)\right.$$

$$\frac{e^{-r(T-t)}}{T}\left\{\int_0^t S(u)\,du + \int_t^T \tilde{E}(S(u)\,|\,\mathcal{F}_t)\,du\right\}$$

$$\frac{e^{-r(T-t)}}{T}\left\{\int_0^t S(u)\,du + \int_t^T e^{ru}\,\tilde{E}(e^{-ru}S_u\,|\,\mathcal{F}_t)\,du\right\}$$

$\cancel{e^{-rT}}\,C^{+}$

$\cancel{e^{r(T-t)}\,S(t)}$

$\cancel{e^{-r(T-u)}\,S(t)}$

$\cancel{e^{-ru}\,S(t)}$

$\cancel{e^{-rt}\,S(t)}$

$$\frac{e^{-r(T-t)}}{T}\left\{\int_0^t S(u)\,du + \int_t^T e^{ru} e^{-rt} S(t)\,du\right\}$$

$$E\left(e^{-ru} S_u \mid \mathcal{F}_t\right) = e^{-rt} S_t$$

$$\frac{e^{-r(T-t)}}{T}\left\{\int_0^t S(u)\,du + e^{-rt} S(t)\int_t^T e^{ru}\,du\right\}$$

$$\frac{e^{-r(T-t)}}{T}\left\{\int_0^t S(u)\,du + e^{-rt}\frac{S(t)}{r} e^{ru}\Big|_t^T\right\}$$

$$\frac{e^{-r(T-t)}}{T}\left\{\int_0^t S(u)\,du + e^{-rt}\frac{S(t)}{r}\left(e^{rT} - e^{+rt}\right)\right\}$$

$$\tilde{E}\left(S_T^2 \mid \mathcal{F}_t\right) = \int_0^\infty \frac{x^2}{\sqrt{2\pi v^2}\,x} e^{-\frac{(\ln x - m)^2}{2v^2}}\,dx$$

where $m = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t)$

$v^2 = \sigma^2(T-t)$

$$\tilde{E}\left(h(S_T) \mid \mathcal{F}_t\right)$$

$$d S_t = r S(t)\,dt + \sigma S(t)\,d\tilde{W}_t$$

$$dS_t^2 = 2S_t\,dS_t + \frac{1}{\cancel{2}} \times \cancel{2}\,(dS_t)^2$$

$$= 2S_t\left(rS(t)\,dt + \sigma S(t)\,d\tilde{W}_t\right) + \sigma^2 S^2(t)\,dt$$

$$dS_t^2 = (2r + \sigma^2)\,S^2(t)\,dt + 2\sigma S^2(t)\,d\tilde{W}_t$$

$$Y_t = S_t^2$$

$$dY_t = (2r + \sigma^2)\,Y(t)\,dt + 2\sigma Y(t)\,d\tilde{W}_t$$

show that $e^{-(2r+\sigma^2)t}\,Y(t)$ is a Martingale

$$\tilde{E}(S_T^2 \mid F_t) = e^{(2r+\sigma^2)T}\,\tilde{E}\left(e^{-(2r+\sigma^2)T}\,S_T^2 \mid F_t\right)$$

$$= e^{(2r+\sigma^2)T}\,e^{-(2r+\sigma^2)t}\,S_t^2$$

4.8 $\quad B(t) = \int_0^t \operatorname{sign}(W_s)\,dW_s$

(a) Prove $B(t)$ is B.M

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \end{cases}$$

clearly $B(t)$ is a Martingale

$$\langle B \rangle_t = \int_0^t (\operatorname{sign}(W_s))^2\,ds$$

$(\text{sign}(W_s))^2 = 1$ with Probability 1

because $P(W_s = 0) = 0$

$$\langle B \rangle_t = \int_0^t 1 \, ds = t$$

Levy $\Rightarrow$ B_t is Brownian Motion.

(b) Are B & W uncorrelated? Are they independent?

$$B(t) = \int_0^t \text{sign}(W_s) \, dW_s$$

X ~~$\oplus$~~ $-X = y$ $\cancel{X \sim N(0,1)}$ $\cancel{\text{cov}(X,-X) = -1}$

$$\begin{pmatrix} X \\ -X \end{pmatrix} \cancel{= \begin{pmatrix} x \\ y \end{pmatrix} = N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)}$$

$B(t)W(t) - \langle B, W \rangle_t$ is a Martingale — (*)

$$\cancel{E(B_t W_t^2)} \; c) \; \langle B, W \rangle_t = \int_0^t \text{sign}(W_s) \, d\langle W, W \rangle_s$$

$$= \int_0^t \text{sign}(W_s) \, ds$$

$$= \int_0^t (\chi_{W_s > 0} \, ds - \chi_{W_s < 0}) \, ds$$

If B & W are uncorrelated then $E(B_t W_t) = 0$

But from (*), we get $E(\langle B, W \rangle_t) = 0$

Suffices to prove $E(\langle B, W \rangle_t) = 0$

$$E(\langle B, W\rangle_t) = \int_0^t \{P(W_s > 0)\, ds - P(W_s < 0)\, ds\}$$

$$= \int_0^t \left(\frac{1}{2} - \frac{1}{2}\right) ds = 0.$$

$\Rightarrow$ B_t & W_t are uncorrelated

$\sigma\{B_s, 0 \le s \le t\}$ & $\sigma(W_s\ 0 \le s \le t\}$

are independent

But B_t & W_t are not independent

$$X_t = \int_0^t \mathbb{1}(W_s > 0)\, dW_s$$

$$Y_t = \int_0^t \mathbb{1}(W_s \le 0)\, dW_s$$

$$\langle X, Y\rangle_t = 0$$

X & Y are not independent

price a security that pays $\frac{S_T}{S_{t_0}}$ at time T, find the

price at $t \le t_0 \le T$

$$\tilde{E}\left(e^{-r(T-t)} \frac{S_T}{S_{t_0}} \,\middle|\, \mathcal{F}_t\right)$$

$$= \tilde{E}\left(\tilde{E}\left(e^{-r(T-t)} \frac{S_T}{S_{t_0}} \,\middle|\, \mathcal{F}_{t_0}\right) \,\middle|\, \mathcal{F}_t\right)$$

$$\tilde{E}\left(\frac{e^{-r(T-t)}}{S_{t_0}}\tilde{E}(S_T \mid \mathcal{F}_{t_0}) \,\Big|\, \mathcal{F}_t\right)$$

$$\tilde{E}\left(\frac{e^{-r(T-t)}}{S_{t_0}} e^{rT}\tilde{E}(e^{-rT}S_T \mid \mathcal{F}_{t_0}) \,\Big|\, \mathcal{F}_t\right)$$

$$\tilde{E}\left(\frac{e^{-r(T-t)}}{\cancel{S_{t_0}}} e^{rT} e^{-rt_0}\cancel{S_{t_0}} \,\Big|\, \mathcal{F}_t\right)$$

$$e^{r(t-t_0)}$$

2) $X_t = \dfrac{W_t}{\sqrt{2(M+t)}}$ $(M>0)$ constant

a) Compute the differential of X_t

$Y_t = f(t, X_t)$

$$dY_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

$$dX_t = -\frac{W_t}{(M+t)^{3/2}\,2^{3/2}}dt$$

$$+\frac{1}{\sqrt{2}\sqrt{M+t}}dW_t + \frac{1}{2}\times 0.$$

$$dX_t = \frac{-W_t}{2^{3/2}(M+t)^{3/2}}dt + \frac{dW_t}{2^{1/2}(M+t)^{1/2}}$$

(b) Find a differential Equation that g must solve so that $g(X_t)$ is a Martingale

$$Y(t) = g(X_t)$$

$$dY(t) = g'(X_t)\,dX_t + \frac{1}{2} g''(X_t)(dX_t)^2$$

$$dY(t) = g'(X_t)\left\{ \frac{-W_t}{2^{3/2}(M+t)^{3/2}}\,dt + \frac{dW_t}{2^{1/2}(M+t)^{1/2}} \right\}$$

$$+ \frac{1}{2} g''(X_t)\frac{dt}{2(M+t)}$$

$$dY(t) = \left\{ \frac{-g'(X_t)\,W_t}{2^{3/2}(M+t)^{3/2}} + \frac{1}{2}\frac{g''(X_t)}{2(M+t)} \right\} dt$$

$$\boxed{W_t = X_t\sqrt{2(M+t)}}$$

$+$ Martingale term

$$dY(t) = \left\{ \frac{-g'(X_t)\,X_t}{2(M+t)} + \frac{1}{4}\frac{g''(X_t)}{(M+t)} \right\} dt + \text{Martingale term}$$

Replacing X_t with x

$$\cancel{\frac{-g'(x)\,x}{2(M+t)} + \frac{1}{4}\frac{g''(x)}{(M+t)}}$$

$$\frac{-g'(x)\,x}{2(M+t)} + \frac{1}{4}\frac{g''(x)}{(M+t)} = 0 \qquad g''(x) = 2g'(x)\,x$$

(c) ~~For the g in the previous part we have that~~

$$\cancel{dg(X_t) = r_t\, dW_t}$$

(1) a) $X_t = W_t^2 B_t$ (W & B are Independent BMs)

Find dX_t

$$Z_t = f(X_t, Y_t)$$

$$dZ_t = \frac{\partial f}{\partial x} dX_t + \frac{\partial f}{\partial y} dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY_t)^2 + \frac{\partial^2 f}{\partial x \partial y} dx\, dy$$

$$dX_t = 2W_t B_t\, dW_t + W_t^2\, dB_t + \frac{1}{\cancel{2}} \times \cancel{2} \times B_t (dW_t)^2 + \frac{1}{2} \times 0 + 0 \quad (dB_t\, dW_t = 0)$$

$$dX_t = B_t\, dt + 2W_t B_t\, dW_t + W_t^2\, dB_t$$

(b) Let $f(x,y)$ be a function. ~~Let~~ Find a condition that f must satisfy s.t $X_t = f(W_t, B_t)$ is a Martingale

$X_t = f(W_t, B_t)$

$$dX_t = \underline{\frac{\partial f}{\partial x} dW_t} + \underline{\frac{\partial f}{\partial y} dB_t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dW_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dB_t)^2 + \frac{\partial^2 f}{\partial x \partial y} dB_t\, dW_t$$

$$= \left\{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(W_t, B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}\right\} dt + \text{Martingale terms}$$

If X_t is a Martingale then f must satisfy

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Find a function $g(y)$ so that $X_t = e^{W_t} g(B_t)$ is a Martingale

$$dX_t = d\,\underline{e^{W_t}}\,\underline{g(B_t)} = g(B_t)\,de^{W_t} + e^{W_t}\,dg(B_t) + (de^{W_t})(dg(B_t))$$

$$dY_t Z_t = Y_t\, dZ_t + Z_t\, dY_t + dY_t\, dZ_t$$

$$de^{W_t} = e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt$$

$$dg(B_t) = g'(B_t)\, dB_t + \frac{1}{2} g''(B_t)\, dt$$

$$dX_t = g(B_t)\left\{e^{W_t} dW_t + \frac{1}{2} e^{W_t} dt\right\} + e^{W_t}\left\{g'(B_t)\, dB_t + \frac{1}{2} g''(B_t)\, dt\right)$$

$$dX_t = e^{W_t}\left(\tfrac{1}{2} g''(B_t) + \tfrac{1}{2} g(B_t)\right) dt + \text{Martingale terms}$$

$$g''(x) + g(x) = 0 \qquad\qquad \frac{\partial^2 y}{\partial x^2} + y(x) = 0$$

$$(D^2 + 1)\, y(x) = 0 \qquad\qquad (D^2 + 1) y = 0$$

$$y(x) = C_1 e^{ix} + C_2 e^{-ix} \qquad\qquad D^2 y = \frac{\partial^2 y}{\partial x^2}$$

$$y(x) = C_1 \cos x + C_2 \sin x \qquad\qquad Dy = \frac{dy}{dx}$$

$$g(x) = C_1 \cos x + C_2 \sin x \qquad\qquad D^3 y = \frac{d^3 y}{dx^3}$$

If M_t is a Martingale

$M_t^2 - \langle M \rangle_t$ is a Martingale

If M_t & N_t are Martingales

$M_t N_t - \langle M; N \rangle_t$ is a Martingale.

When is the product of Martingales a Martingale?

$M_t N_t$ is a Martingale $\Leftrightarrow$ $\langle M, N \rangle_t = 0$

4.4. $d\tilde{P} = e^{\alpha X + \beta}\, dP.$

$\tilde{P}(A) = \int_A e^{\alpha X + \beta}\, dP$

$\tilde{E}(Y) = E(Y Z_T)$

$E(Y) = \tilde{E}\left(\frac{Y}{Z_T}\right)$

$\tilde{P}(A) = \int_A Z_T\, dP$

$X \sim N(0,1)$ under P, a, α & $\beta \in R$

$X + a \sim N(0,1)$ under $\tilde{P}$

$E^P(e^{\alpha X + \beta}) = 1$

$e^{\beta} E^P(e^{\alpha X}) = 1$

$e^{\beta} e^{\frac{\alpha^2}{2}} = 1$ α & β

$\frac{\alpha^2}{2} + \beta = 0$ $\left(\frac{a^2}{2} - \frac{a^2}{2} = 0\right)$

$X + a \sim N(0,1)$ under $\tilde{P}$

$\tilde{P}((X+a) \in (y, y+dy)) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$

$\tilde{P}((X+a) \in (y, y+dy))$

~~$P(A) = \int_A e^{-\alpha X - \beta}\, d\tilde{P}$~~

$$\tilde{P}((X+a)\in(y,y+dy)) = \int_{\{X+a\in y,\, y+dy\}} e^{\alpha X+\beta}\, dP$$

$$= E^{P}\left(e^{\alpha X+\beta}\, 1((X+a)\in(y,y+dy))\right)$$

$$= \int_{-\infty}^{\infty} e^{\alpha x+\beta}\, 1(x+a\in(y,y+dy))\, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \int_{-\infty}^{\infty} e^{\alpha x+\beta}\, 1(x\in(y-a,y-a+dy))\, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \int_{y-a}^{y-a+dy} e^{\alpha x+\beta}\, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$\cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{y^2}{2}}\, \cancel{dy} = e^{\alpha(y-a)+\beta}\, \cancel{\frac{1}{\sqrt{2\pi}}}\, e^{-\frac{(y-a)^2}{2}}\, \cancel{dy}$$

$$1\ \cancel{e^{-y^2/2}} = e^{\alpha y-\alpha a+\beta}\, \cancel{e^{-\frac{y^2}{2}}}\, e^{-\frac{a^2}{2}}\, e^{+ya}$$

$$1 = e^{(\alpha+a)y}\, e^{-\alpha a+\beta-\frac{a^2}{2}}$$

$$\alpha+a=0 \qquad \alpha=-a$$

$$-\alpha a+\beta-\frac{a^2}{2}=0$$

$$a^2+\beta-\frac{a^2}{2}=0 \qquad \beta=-\frac{a^2}{2} \qquad \alpha=-a$$