STOCHASTIC CALCULUS PROBLEM SOLVING SESSION

$dS_t = \mu S_t dt + \sigma S(t) dW_t$ under P

$v(T, S_T) = h(S_T)$

For European call $h(S_T) = (S_T - K)^+$

If $v(t, S_t)$ is the price of the derivative security at time t

$$v(t, S_t) = \tilde{E}\left(e^{-r(T-t)} h(S_T) \mid \mathcal{F}_t\right)$$

$$dS_t = rS(t)dt + \sigma S(t) d\tilde{W}_t$$

$$\text{Portfolio } (\pi) = v(t, S_t) + \Delta(t) S(t)$$

$$\begin{aligned}
d\pi &= dv(t, S_t) + \Delta(t) dS(t) \\
&= \frac{\partial v}{\partial t}(t, S_t) dt + \frac{\partial v}{\partial s} dS_t + \frac{1}{2}\frac{\partial^2 v}{\partial s^2}(dS_t)^2 + \Delta(t) dS(t) \\
&= \left\{\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s}\mu S(t) + \frac{1}{2}\frac{\partial^2 v}{\partial s^2}\sigma^2 S^2(t) + \Delta(t)\mu S(t)\right\} dt \\
&\quad + \left\{\frac{\partial v}{\partial s}\sigma S(t) + \Delta(t)\sigma S(t)\right\} dW_t
\end{aligned}$$

$$\Delta(t)\sigma S(t) + \frac{\partial v}{\partial s}(t, S(t))\sigma S(t) = 0$$

$$\Rightarrow \Delta(t) = -\frac{\partial v}{\partial s}(t, S(t))$$

$$\begin{aligned}
d\pi &= r\pi dt \\
&= r\left(v(t, S(t)) - \frac{\partial v}{\partial s}(t, S(t)) S(t)\right) dt
\end{aligned}$$

②

$$d\pi = \left\{ \frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} \mu s(t) + \frac{1}{2}\frac{\partial^2 v}{\partial s^2}\sigma^2 s^2(t) - \frac{\partial v}{\partial s}\mu s(t) \right\} dt$$

$$= r\left(v(t, s(t)) - \frac{\partial v}{\partial s}(t, s(t))\, s(t) \right) dt$$

$$\Rightarrow \frac{\partial v}{\partial t} + r s(t) \frac{\partial v}{\partial s}(t, s(t)) + \frac{1}{2}\sigma^2 s^2(t) \frac{\partial^2 v}{\partial s^2} = r v(t, s(t))$$

$$v(T, S_T) = h(S_T)$$

$$\frac{\partial F}{\partial t}(t,x,y) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\delta^2 \frac{\partial^2 F}{\partial y^2}(t,x,y) = 0$$

$$F(T,x,y) = xy \quad , \quad F(t,x,y)$$

$$dX_t = \sigma\, dW_t^1$$

$$dX_t = a\,dt + b\,dW_t^1 \qquad W_t^1 \perp\!\!\!\perp W_t^2 \qquad dY_t = \delta\, dW_t^2$$

$$dY_t = r\,dt + s\,dW_t^2$$

$$dF(t, X_t, Y_t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX_t + \frac{\partial F}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(dY_t)^2 + \frac{\partial^2 F}{\partial x \partial y} dX_t\, dY_t$$

$$= \left\{ \frac{\partial F}{\partial t} + a\frac{\partial F}{\partial x}(t, X_t, Y_t) + r\frac{\partial F}{\partial y}(t, X_t, Y_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} b^2 + \frac{1}{2}\frac{\partial^2 F}{\partial y^2} s^2 \right\} dt + \text{Martingale terms}$$

$$a = 0, \quad b = \sigma, \quad r = 0, \quad s = \delta$$

$$dF(t,X_t,Y_t) = \left\{ \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,X_t,Y_t) + \frac{1}{2}\delta^2 \frac{\partial^2 F}{\partial y^2}(t,X_t,Y_t) \right\} dt$$ 3

+ Martingale terms

$$dF(t,X_t,Y_t) = \_\_\, dW_t^1 + \_\_\, dW_t^2$$

$$F(T,X_T,Y_T) - F(t,X_t,Y_t) = \int_t^T \_\_\, dW_s^1 + \int_t^T \_\_\, dW_s^2$$

$$F(t,X_t,Y_t) = E\left(F(T,X_T,Y_T) \mid \mathcal{F}_t \right)$$

$$F(t,X_t,Y_t) = E\left(X(T)\,Y(T) \mid \mathcal{F}_t \right) = X(t)\,Y(t)$$

$$dX_t = \sigma\, dW_t^1 \qquad \cancel{X_t - X_0 = \int_0^t \sigma\, dW_t^1 \Rightarrow X_t \sim N(X_0, \sigma^2 t)}$$

$$dY_t = \delta\, dW_t^2$$

$$\cancel{d\,X(t)Y(t) = X(t)\,dY(t) + Y(t)\,dX(t) + dX(t)\,dY(t)}$$

$$= \cancel{Y(t)}$$

$$X_T - X_t = \int_t^T \sigma\, dW_s^1 \Rightarrow X_T \sim N(X_t, \sigma^2 (T-t))$$

$$Y(T) \sim N(Y(t), \delta^2 (T-t))$$

X & Y are independent

$$X_T = X_t + \sigma (W_T^1 - W_t^1)$$

$$Y_T = Y_t + \delta (W_T^2 - W_t^2)$$

$$\frac{\partial F}{\partial t} + \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2} + = -rF$$

$$F(T,x) = \Phi(x)$$

$$dX(t) = a\,dt + b\,dW(t)$$

$$dF(t,X_t) = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dX_t)^2$$

$$= \left(\frac{\partial F}{\partial t} + a\frac{\partial F}{\partial x} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial x^2}\right)dt + \underline{\text{Martingale}}\ \text{term}$$

$$a = \mu(t,X_t) \qquad b = \sigma(t,X(t))$$

$$dX(t) = \cancel{a}\,\mu(t,X_t)\,dt + \sigma(t,X(t))\,dW(t)$$

$$dF(t,X_t) = -rF(t,X_t)\,dt + \sigma(t,X(t))\frac{\partial F}{\partial x}(t,X(t))\,dW_t$$

$$d\,e^{rt}F(t,X_t) = e^{rt}\,dF(t,X_t) + re^{rt}\,dt\,F(t,X(t)) + \underbrace{dF(t,X(t))\,de^{+rt}}_{0}$$

$$d\,e^{rt}F(t,X_t) = e^{rt}\left(\underline{-rF(t,X_t)}\,dt + \sigma(t,X(t))\frac{\partial F}{\partial x}(t,X(t))\,dW_t\right) + \underline{re^{rt}}F(t,X(t))\,dt$$

$$= e^{rt}\sigma(t,X(t))\frac{\partial F}{\partial x}\,dW(t)$$

$$e^{rT}F(T,X_T) - e^{rt}F(t,X_t) = \int_t^T e^{rs}\sigma(t,X(s))\frac{\partial F}{\partial x}\,dW(s)$$

Taking conditional Expectation on both sides, we get (5)

$$e^{rt} F(t, X_t) = E\left(e^{rT} F(T, X_T) \mid \mathcal{F}_t \right)$$

$$F(t, X_t) = E\left(e^{r(T-t)} \phi(X_T) \mid \mathcal{F}_t \right)$$

Black Scholes PDE

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

$V(T, S) = (S-K)^+$ or $h(S)$ more generally

$$\cancel{dS(t)}\; dS(t) = a\,dt + b\,d\tilde{W}_t$$

$$dV(t, S(t)) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS(t) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2$$

$$= \left\{ \frac{\partial V}{\partial t} + \cancel{rs}\, a \frac{\partial V}{\partial S} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial S^2} \right\} dt$$

$$+ \frac{\partial V}{\partial S} b\, d\tilde{W}_t$$

$$a = rS(t) \qquad b = \sigma S(t)$$

$$dV(t, S(t)) = \left\{ \frac{\partial V}{\partial t} + rS(t)\frac{\partial V}{\partial S}(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 V}{\partial S^2}(t, S(t)) \right\} dt$$

$$+ \frac{\partial V}{\partial S} \sigma S(t)\, d\tilde{W}_t$$

$$= rV(t, S(t))\, dt + \frac{\partial V}{\partial S}\sigma S(t)\, d\tilde{W}_t$$

$$d e^{-rt} V(t, S(t)) = -re^{-rt} V(t, S(t))\, dt + e^{-rt} dV(t, S(t))$$

$$= e^{-rt} \sigma \cancel{s}\, S(t) \frac{\partial V}{\partial S}(t, S(t))\, d\tilde{W}_t$$

$$e^{-rT} V(T, S(T)) - e^{-rt} V(t, S(t)) = \int_t^T e^{-ru} \sigma S(u) \frac{\partial V}{\partial s}(t, S(u)) \, dW_u$$

Taking conditional Expectation on both sides we get

$$e^{-rt} V(t, S(t)) = \tilde{E}\left(e^{-rT} V(T, S(T)) \mid \mathcal{F}_t\right)$$

$$V(t, S(t)) = \tilde{E}\left(e^{-r(T-t)} V(T, S(T)) \mid \mathcal{F}_t\right)$$

$$V(t, S(t)) = \tilde{E}\left(e^{-r(T-t)} h(S(T)) \mid \mathcal{F}_t\right)$$

$$dS(t) = rS(t)\,dt + \sigma S(t)\, d\tilde{W}(t)$$

$$d\ln S_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma \, d\tilde{W}(t)$$

$$\Rightarrow \ln S_T = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}(T) - \tilde{W}(t)\right)$$

$$\Rightarrow \ln S_T \sim N(m, v^2)$$

where $m = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t)$

$v^2 = \sigma^2 (T-t)$

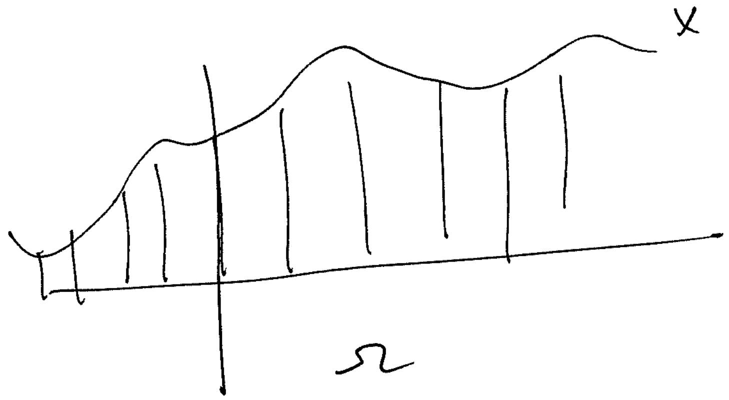
$$P(S_T = x, x+dx) = \frac{1}{\sqrt{2\pi v^2}\, x} e^{-\frac{(\ln x - m)^2}{2v^2}}$$

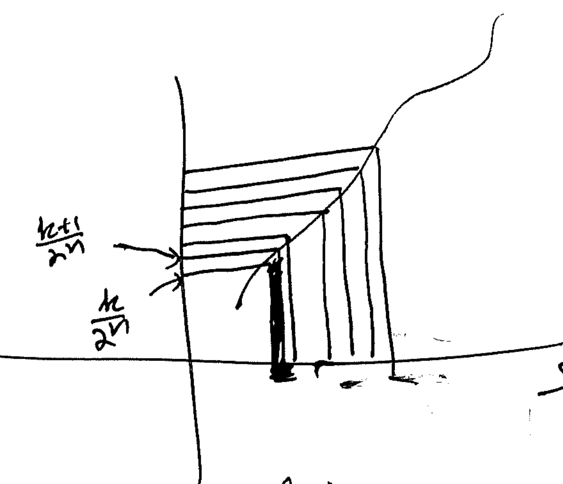
$$V(t, S(t)) = \int_0^\infty e^{-r(T-t)} h(x) \frac{1}{\sqrt{2\pi v^2}\, x} e^{-\frac{(\ln x - m)^2}{2v^2}} \, dx$$

Girsanov's theorem & change of Measure

change of Measure:

$E(X) = \int X \, dP \qquad X : \Omega \to \mathbb{R}$





$\Omega, P \qquad P(\Omega) = 1$

$P(\Omega)$ Given any set (almost) in Ω

$\mu(E) \quad P(E) \quad Q(E)$

$E = \{ \omega \in \Omega \;\; f(\omega) \in [\frac{k}{2^n}, \frac{k+1}{2^n}) \}$

$\frac{k}{2^n} P(E)$

Probability Measures:

If you have a +ve Random variable whose $E^P(X) = 1$

$\int_\Omega X \, dP = 1 \qquad Q(A) = \int_A X \, dP \quad A \subseteq \Omega$

$Q(\Omega) = \int_\Omega X \, dP = E^P(X) = 1$

$Q(A) = \int X \, dP \qquad X \geq 0$ with $E^P(X) = 1$

$dQ = X \, dP$

$\frac{dQ}{dP} = X$

$X \to$ Radon Nikodym derivative

$Z_t = e^{M_t - \frac{\langle M \rangle_t}{2}} \qquad M_0 = 0$

$Z_t = e^{\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds}$

$E(Z_t) = 1$

Fix: $T \quad E^P(Z_T) = 1$

$Q(A) = \int_A Z_T \, dP \qquad Q(\Omega) = \int_\Omega Z_T \, dP = E^P(Z_T) = 1$

$dQ = Z_T \, dP$

$$E^Q(Y) = \int_\Omega Y \, dQ$$

$$= \int Y Z_T \, dP$$

$$= E^P(Y Z_T)$$

$E^Q(Y) = E^P(Y Z_T)$

$E^P(Y) = E^Q\left(\frac{Y}{Z_T}\right)$

$$E^{Q}\left(\frac{Y}{Z_T}\right) = \int_{\Omega} \frac{Y}{Z_T}\, dQ \qquad dQ$$

$$= \int_{\Omega} \frac{Y}{\cancel{Z_T}} \cancel{Z_T}\, dP$$

$$= E^{P}(Y)$$

Girsanov's theorem:

$d\tilde{W}_t = \theta(t)\, dt + dW_t$ $\quad W_t$ is P Brownian Motion

$\tilde{W}_t$ is not a Brownian Motion under P

$$Q(A) = \int_A Z_T\, dP$$

where $Z_t = e^{\int_0^t -\theta(s)\, dW_s - \frac{1}{2}\int_0^t \theta^2(s)\, ds}$

$$Z_T = e^{-\int_0^T \theta(s)\, dW_s - \frac{1}{2}\int_0^T \theta^2(s)\, ds}$$

Under Q

$\tilde{W}_t$ is a Brownian Motion for $0 \le t \le T$

$(\Omega, \mathcal{F}, P)$ $\mathcal{F}_t$

$\boxed{(\Omega, \mathcal{F}, Q)}$ $\mathcal{F}_T$

$E^{Q}(Y) = E^{P}(Y Z_T)$ $\quad dQ = Z_T \, dP$

$E^{P}(Y) = E^{Q}\left(\frac{Y}{Z_T}\right)$

$\quad\quad \|$

$$\frac{dQ}{dP} = Z_T \Rightarrow \frac{dP}{dQ} = \frac{1}{Z_T}$$

$$\int Y \frac{dP}{dQ} \, dQ = \int \frac{Y}{Z_T} \, dQ$$

Bayes theorem :

given any Random variable X which is $\mathcal{F}_T$ measurable.

$$\left.\begin{aligned} E^{Q}(X \mid \mathcal{F}_t) &= \frac{1}{Z_t} E^{P}(X Z_T \mid \mathcal{F}_t) \\ E^{P}(X \mid \mathcal{F}_t) &= Z_t E^{Q}\left(\frac{X}{Z_T} \,\Big|\, \mathcal{F}_t\right) \end{aligned}\right\} \text{Baye's Rules}$$

~~Q~~ $\tilde{M}_t$ is a Martingale under $Q \iff \tilde{M}_t Z_t$ is a P Martingale