Last Time: Joint QV.

$X, Y \longrightarrow$ two processes.

$P =$ partion of $[0, T]$. $P = \{\overset{t_0}{\overset{\shortparallel}{0}}, t_1, t_2 \dots, \overset{t_n}{\overset{\shortparallel}{T}}\}$.

$$\lim_{\|P\| \to 0} \sum \underbrace{(X(t_{i+1}) - X(t_i))}_{\text{size } \sqrt{t_{i+1} - t_i}} \underbrace{(\cancel{X}\, Y(t_{i+1}) - Y(t_i))}_{\text{size } \sqrt{t_{i+1} - t_i}} \longrightarrow [X, Y](T).$$

$$4ab = (a+b)^2 - (a-b)^2 \quad \dots \quad (*)$$

<u>Last time</u>: $[X, Y] = \frac{1}{4}\left([(X+Y), (X+Y)] - [(X-Y), X-Y]\right)$.

Product Rule: $d(XY) = X\,dY + Y\,dX + d[X, Y]$.

Prop: $X_1, X_2 \longrightarrow$ two Itô Processes.

$\Delta_1, \Delta_2 \longrightarrow$ two adapted processes.

$$I_1(t) = \int_0^t \Delta_1(s)\, dX_1(s) \quad \& \quad I_2(t) = \int_0^t \Delta_2(s)\, dX_2(s).$$

Then $$[I_1, I_2](t) = \int_0^t \Delta_1(s)\, \Delta_2(s)\, \cancel{ds}\, d[X_1, X_2](s).$$

i.e. $$d[I_1, I_2](t) = \Delta_1(t)\, \Delta_2(t)\, d[X_1, X_2](t).$$

Pf: Follows from $(*)$.

~~Prop~~ Cor: X, Y two Ito processes.

B a adapted process of Bounded variation.
(finite first var).

Then $[X, B] = 0$

Pf: $[X \pm B, X \pm B] = [X, X]$.

$\Rightarrow 4[X, B] = [X+B, X+B] - [X-B, X-B] = 0$

QED.

Multi D Itô (Doeblin):

Let $X_1, X_2 \dots X_n$ be n Itô processes.

$f : [0,\infty) \times \mathbb{R}^n$ $\quad$ f is C^1 in t

↳ $f = f(t, x_1, x_2, \dots x_n)$. & C^2 in x_i $\forall i$.

~~Then $f(X$ $f(T, X$~~ $\quad X = (X_1, X_2 \dots X_n)$.

$x = (x_1, x_2, \dots x_n)$.

$$f(T, X(T)) = f(0, X(0)) + \int_0^T \partial_t f(t, X(t))\, dt$$

$$+ \sum_{i=1}^{n} \int_0^T \partial_i f(t, X(t))\, \cancel{dt}\, dX_i(t) + \frac{1}{2} \sum \int_0^T \partial_i \partial_j f(t, X(t))\, d[X_i, X_j](t).$$

Differential form. ~~Cont~~ Cont. $(t, X(t))$ everywhere.

$$d\big(f(t,X(t))\big) = \partial_t f \, dt + \sum_{i=1}^{n} \partial_i f \, dX_i(t)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i \partial_j f \, d[X_i, X_j](t)$$

Intuition: $P = \{ \overset{t_0}{\overset{\shortparallel}{0}} < t_1 < \cdots \, t_n = T \}$

$$\cancel{f(t, X} \; f(T, X(T)) - f(0, X(0)) = \sum f(t_{i+1}, \xi_{i+1}) - f(t_i, \xi_i).$$

$$\xi_i = X(t_i).$$

Taylor expand each term $f(t_{i+1}, \xi_{i+1}) - f(t_i, \xi_i)$.

<u>Notation:</u> $\cancel{\Delta_i X =}$

2D Ito: $f = f(x, y)$. X Y Ito processes.

$$\xi_i = (X(t_i), Y(t_i)).$$

$$\Delta_i X = X(t_{i+1}) - X(t_i).$$

$$\Delta_i Y = \cancel{X} Y(t_{i+1}) - Y(t_i).$$

$$f(X(T), Y(T)) - f(X(0), \cancel{X} Y(0)) = \sum f(\xi_{i+1}) - f(\xi_i).$$

$\underline{\text{Taylor:}}$

$$f(\xi_{i+1}) - f(\xi_i) = \partial_x f(\xi_i) . \Delta_i X + \partial_y f \, \Delta_i Y$$

$$+ \frac{1}{2}\Big(\partial_x^2 f(\xi_i)(\Delta_i X)^2 + \partial_y^2 f(\xi_i) . (\Delta_i Y)^2 .$$

$$+ 2 \partial_x \partial_y f(\xi_i) . (\Delta_i X)(\Delta_i Y) \Big).$$

$$0 \longleftarrow \boxed{+ \text{ third order terms}}$$

Expect $\sum_i \partial_x f(\xi_i)\,\Delta_i X \longrightarrow \int_0^T \partial_x f \, dX.$

lly $\sum \partial_y f(\xi_i)\,\Delta_i Y \longrightarrow \int_0^T \partial_y f \, dY.$

$\sum \partial_x^2 f \,(\Delta_i X)^2 \longrightarrow \int_0^T \partial_x^2 f \; d[X,X].$

$\sum \partial_y^2 f \,(\Delta_i Y)^2 \longrightarrow \int_0^T \partial_y^2 f \; d[Y,Y].$

4ab trick

$\Rightarrow \sum \partial_x \partial_y f \,(\Delta_i X)(\Delta_i Y) \longrightarrow \int_0^T \partial_x \partial_y f \, d[X,Y].$

2D Ito:

$$df(X,Y) = \cancel{\partial_t f}\, dt \xrightarrow{} 0 \; + \partial_x f\, dX + \partial_y f\, dY$$

$$+ \frac{1}{2}\left(\partial_x^2 f\, d[X,X]. + \partial_y^2 f\, d[Y,Y]. + 2\partial_x\partial_y f\, d[X,Y].\right).$$

~~Most~~ Common use: W_1 & W_2 2 Independent BM.

$$d f(W_1(t), W_2(t)). = \partial_x f\, dW_1 + \partial_y f\, dW_2 + \frac{1}{2}\left(\partial_x^2 f\, dt + \partial_y^2 f\, dt + 2\partial_x\partial_y f\, d[W_1, W_2].\right).$$

Claim: $d[W_1, W_2] = 0$.

$\boxed{\begin{array}{l}\text{More generally: If } M \text{ \& } N \text{ are 2 mg's (wrt } \{\mathcal{F}_t\}) \\ \text{If } M \text{ is ind of } N \text{ then } [M,N] = 0.\end{array}}$

Note $[M,N] = 0 \not\Longrightarrow M$ ind of N.

Prop: Say M, N are 2 mg's (cts) wrt $\{\mathcal{F}_t\}$.

$(E\,M(t)^2 < \infty, \; E\,N(t)^2 < \infty)$.

Then ① $MN - [M,N]$ is a mg wrt $\{\mathcal{F}_t\}$.

& ② If B is cts, adapted, BV & $MN - B$ is a mg $\Rightarrow B = [M,N]$

Proof I: 4ab trick & $M^2 - [M,M]$ is a mg.

Proof II: Product rule:

$$d(MN) = \underbrace{M\,dN}_{\substack{mg \\ (\because N \text{ is a Mg})}} + \underbrace{N\,dM}_{\substack{mg \\ (\because M \text{ is mg})}} + d[M,N].$$

$$\Rightarrow \cancel{MN}\; M(T)N(T) - M(0)N(0) - [M,N](T)$$

$$= \int_0^T M(t)\,dN + \int_0^T N(t)\,dM \rightarrow \text{a mg}.$$

Cor: If M, N are 2 mg's wrt $\{\mathcal{F}_t\}$.

then M ind of $N \implies [M, N] = 0$

Pf: By Prop, only NTS MN is a mg.

ie. NTS $E\left(M(t)N(t) \mid \mathcal{F}_s\right) = M(s)N(s)$.

M, N ind. $E(MN) = EM\, EN$.

~~Expect~~ $E\left(M(t)N(t) \mid \mathcal{F}_s\right) \neq \underbrace{E\left(M(t) \mid \mathcal{F}_s\right)}_{M(s)} \underbrace{E\left(N(t) \mid \mathcal{F}_s\right)}_{N(s)}$.

WRONG.

Correct Proof:

$$E\big(M(t)N(t) \mid \mathcal{F}_s\big) = E\Big(\big((M(t)-M(s))+M(s)\big)\big((N(t)-N(s))+N(s)\big) \,\Big|\, \mathcal{F}_s\Big)$$

$$= M(s)N(s) + \underbrace{E\big((M(t)-M(s))\,N(s) \mid \mathcal{F}_s\big)}_{0}$$

IOU. $$\boxed{+\,E\big((M(t)-M(s))(N(t)-N(s)) \mid \mathcal{F}_s\big)}$$ also 0 (ind) (IOU).

$$+\,\underbrace{E\big(M(s)\,(N(t)-N(s)) \mid \mathcal{F}_s\big)}_{0}.$$

$$\Rightarrow E\big(M(t)N(t) \mid \mathcal{F}_s\big) = M(s)N(s).$$

$$E\Big(\big(M(t)-M(s)\big)N(t)\,\Big|\,\mathcal{F}_s\Big) = E\big(M(t)-M(s)\big)\cdot E\big(N(t)\,|\,\mathcal{F}_s\big).$$

$$E\Big(E\Big(\big(M(t)-M(s)\big)N(t)\,\Big|\,\mathcal{F}_t\Big)\,\Big|\,\mathcal{F}_s\Big). \quad \boxed{\text{IOU}}$$

Eg: $M(t) = \int_0^t \mathbb{1}_{\{W(s)<0\}}\, dW(s).$

$N(t) = \int_0^t \mathbb{1}_{\{W(s)\geq 0\}}\, dW(s).$

$[M,N] = \int_0^t 0\, ds = 0.$ But M & N are not ind!

$$M(t) + N(t) = \int_0^t 1 \, dW = W(t).$$

$$M(t) + N(t) = W(t).$$

$\boxed{\text{Compute } E\left(M(t)^2 N(t)^2\right)}$ $\neq$ You do.

$$\cancel{\Rightarrow M(t)^2 + N(t)^2 + 2M(t)N(t) = W(t)^2.}$$

$$\cancel{E(\qquad) = E\,W(t)^2.}$$

$$\cancel{E \quad 2\,E\,M(t)N(t). = t}$$

Def: We say W is a d-dimensional BM.

if ① $W = (W_1, W_2, \dots W_d)$.

② Each W_i is a B.M. (1D).

③ for $i \neq j$ W_i is ind of W_j.

Filtration:

Define $\mathcal{F}_t^W = \sigma\left(\bigcup_{\substack{s \le t \\ i \in \{1, \dots d\}}} \sigma(W_i(s)) \right)$.

You check: Each W_i is a mg wrt $\mathcal{F}_t^W$.

In fact $W_i(t) - W_i(s)$ is independent of $\mathcal{F}_s^W$

Since each coordinate is independent:

$$d[W_i, W_j](t) = \begin{cases} 0 & i \neq j \\ dt. & i = j \end{cases}$$

$$\boxed{d[W_i, W_j](t) = \mathbb{1}_{\{i=j\}}\, dt}.$$

Hence if W is a d-dim BM.

f is C^1 in t & C^2 in x.

Then $d f(t, W(t)) = \partial_t f \, dt + \partial_i f \, dW_i(t)$

$$+ \frac{1}{2} \boxed{\sum_{i=1}^{d} \partial_i^2 f} \, dt$$

Note $\sum_{i=1}^{d} \partial_i^2 f =$ Laplacian of f. $= \Delta f$.

$\nabla^2 f$.

$\partial_x^2 f + \partial_y^2 f + \partial_z^2 f$