STOCH CAL PROBLEM SOLVING SESSION

$$\frac{\partial v}{\partial t} + \frac{\partial v}{\partial s} r s + \frac{1}{2}\frac{\partial^2 v}{\partial s^2}\sigma^2 s^2 = r v$$

$$v(T, S_T) = (S_T - K)^+ ; \; h(S_T)$$

$$v(t, S_t) = \tilde{E}\left(e^{-r(T-t)}(S_T - K)^+ \mid \mathcal{F}_t\right)$$

$$\left.\begin{aligned} &\frac{\partial h(t,x)}{\partial t} + a\frac{\partial h(t,x)}{\partial x} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2} b^2(t,x) = 0 \\ &h(T, x) = g(x) \end{aligned}\right\}$$

$$dX_t = a\,dt + b\,dW_t$$

$$d\,h(t, X_t) = \left(\underbrace{\frac{\partial h}{\partial t}(t, X_t) + \frac{\partial h}{\partial x}(t, X_t)\,a}_{} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(t, X_t)\,b^2\right)dt$$

$$+ b\frac{\partial h}{\partial x}(t, X_t)\,dW_t$$

$$d\,h(t, X_t) = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}(dX_t)^2$$

$$h(T, X_T) - h(t, X_t) = \int_t^T b\frac{\partial h}{\partial x}(t, X_s)\,dW_s$$

$$g(X_T) - h(t, X_t) = \int_t^T b\frac{\partial h}{\partial x}(t, X_s)\,dW_s$$

$$E\left(g(X_T) \mid \mathcal{F}_t\right) - E\left(h(t, X_t) \mid \mathcal{F}_t\right) = \underbrace{E\left(\int_t^T b\frac{\partial h}{\partial x}(t, X_s)\,dW_s \,\Big|\, \mathcal{F}_t\right)}_{=0}$$

$h(t, X_t) = E(g(X_T) \mid \mathcal{F}_t)$

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$dS_t = \mu S_t dt + \sigma S_t dW_t$

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

$V(T, S_T) = h(S_T) \qquad V(t, S_t)$

$dS_t = rS(t)dt + \sigma S(t) d\tilde{W}_t$

$$dV(t, S_t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2$$

$$= \left\{\frac{\partial V}{\partial t} + rS_t \frac{\partial V(t, S_t)}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t)\right\} dt$$

$$+ \sigma \frac{\partial V}{\partial S}(t, S_t) S(t) d\tilde{W}_t$$

$$dV(t, S_t) = rV(t, S_t)dt + \sigma \frac{\partial V}{\partial S}(t, S_t) S(t) d\tilde{W}_t$$

$de^{-rt} V(t, S_t)$

$dX_t Y_t = X_t dY_t + Y_t dX_t + dX_t dY_t$

$$de^{-rt} V(t, S_t) = -re^{-rt} dt\, V(t, S_t) + e^{-rt} dV(t, S_t)$$

$$\underbrace{(-re^{-rt}dt)\, dV(t, S_t)}_{0}$$

$$d e^{-rt} v(t,S_t) = -r v(t,S_t) e^{-rt} dt + e^{-rt} \left\{ \left(\frac{\partial v}{\partial t} + r S_t \frac{\partial v}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial s^2} \right) dt + \sigma S(t) \frac{\partial v}{\partial s} d\tilde{W}_t \right\}$$

$$d e^{-rt} v(t,S_t) = e^{-rt} \left\{ \left(\frac{\partial v}{\partial t} + r S_t \frac{\partial v}{\partial s} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 v}{\partial s^2} - r v \right) dt + \sigma S(t) \frac{\partial v}{\partial s} d\tilde{W}_t \right\}$$

$$d e^{-rt} v(t,S_t) = e^{-rt} \sigma S(t) \frac{\partial v}{\partial s} d\tilde{W}_t$$

$$e^{-rT} v(T,S_T) - e^{-rt} v(t,S_t) = \int_t^T e^{-ru} \sigma S(u) \frac{\partial v}{\partial s}(u, S_u) \, d\tilde{W}_u$$

$$e^{-rt} v(t,S_t) = \tilde{E}\left(e^{-rT} v(T,S_T) \mid \mathcal{F}_t\right)$$

$$e^{-rt} v(t,S_t) = \tilde{E}\left(e^{-rT} h(S_T) \mid \mathcal{F}_t\right)$$

$$v(t,S_t) = \tilde{E}\left(e^{-r(T-t)} h(S_T) \mid \mathcal{F}_t\right)$$

$$v(t, S_t) = \tilde{E}\left(e^{-r(T-t)} h(S_T) \mid \mathcal{F}_t\right)$$

$$v(t, S_t) = \tilde{E}\left(e^{-r(T-t)} (S_T - K)^+ \mid \mathcal{F}_t\right)$$

$$dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$$

$$d\ln S_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2}(dS_t)^2$$

$$= \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\,d\tilde{W}_t$$

$$\ln S_T - \ln S_t = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)$$

$$\ln S_T = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\left(\tilde{W}_T - \tilde{W}_t\right)$$

$$X \sim N(\mu, \sigma^2)$$

$y \sim e^X$ $\quad$ $y \sim$ lognormally distributed $\quad$ $\ln y \sim N(\mu, \sigma^2)$

$$P(y = y) = \frac{1}{\sqrt{2\pi\sigma^2}\,|y|} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}$$

$$P(y = (y, y+dy)) = \frac{1}{\boxed{\sqrt{2\pi\sigma^2}\,y}} e^{-\frac{(\ln y - \mu)^2}{2\sigma^2}}\,dy$$

Conditioned on $\mathcal{F}_t$

$$\ln S_T \sim N(m, v^2)$$

$$m = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

$$v^2 = \sigma^2 (T-t)$$

$$P(S_T = y) = \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}}$$

$$V(t, S_t) = \tilde{E}\left(e^{-r(T-t)}(S_T - K)^+ \mid \mathcal{F}_t\right)$$

$$= \int_0^\infty e^{-r(T-t)} (y-K)^+ \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}} \, dy$$

$$= \int_K^\infty e^{-r(T-t)} (y-K) \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}} \, dy$$

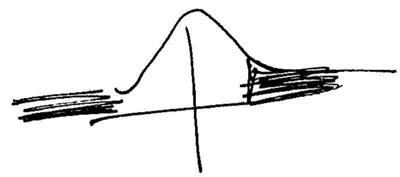
$$\int_K^\infty e^{-r(T-t)} y \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}} \, dy \quad - (1)$$

$$- K \int_K^\infty e^{-r(T-t)} \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}} \, dy \quad - (2)$$

2 $$K\int_K^\infty e^{-r(T-t)} \frac{1}{\sqrt{2\pi v^2}\, y}\, e^{-\frac{(\ln y - m)^2}{2v^2}}\, dy$$

$$z = \frac{\ln y - m}{v} \qquad v\,dz = \frac{1}{y}\,dy \Rightarrow dy = yv\,dz$$

$$K\int_{\frac{\ln K - m}{v}}^\infty e^{-r(T-t)} \frac{1}{\sqrt{2\pi}\, \cancel{vy}}\, e^{-\frac{z^2}{2}}\, dz\, \cancel{yv}$$



$$= Ke^{-r(T-t)} \int_{\frac{\ln K - m}{v}}^\infty \frac{1}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, dz$$

$$= Ke^{-r(T-t)}\, N\left(\frac{m - \ln K}{v}\right)$$

$$m = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

$$v = \sigma\sqrt{T-t}$$

$$= Ke^{-r(T-t)}\, N\left(\frac{\ln\frac{S_t}{K} + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

(1) $$e^{-r(T-t)} \int_{K}^{\infty} y \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}} dy$$

(7)

$$z = \frac{\ln y - m}{v} \quad dy = y v\, dz \quad \Rightarrow y = e^{m+zv}$$

$$e^{-r(T-t)} \int_{\frac{\ln K - m}{v}}^{\infty} y \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$e^{-r(T-t)} \int_{\frac{\ln K - m}{v}}^{\infty} e^{m+zv} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$e^{m} e^{-r(T-t)} \int_{\frac{\ln K - m}{v}}^{\infty} e^{zv} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz$$

$$e^{m} e^{-r(T-t)} \int_{\frac{\ln K - m}{v}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-v)^2}{2}} e^{\frac{v^2}{2}} dz$$

$$e^{m+\frac{v^2}{2}} e^{-r(T-t)} \int_{\frac{\ln K - m}{v}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-v)^2}{2}} dz$$

$$x = \frac{z-v}{1}$$
$$dx = dz$$

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$$e^{m+\frac{v^2}{2}} e^{-r(T-t)} \int_{\frac{\ln k - m - v^2}{v}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$e^{m+\frac{v^2}{2}} e^{-r(T-t)} N\left(\frac{m + v^2 - \ln k}{v} \right)$$

$$m = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

$$v = \sigma\sqrt{T-t}$$

$$= S_t N\left(\frac{\ln S_t + \sigma^2(T-t) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$= S_t N\left(\frac{\ln \frac{S_t}{k} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \right)$$

$$\tilde{E}\left(e^{-r(T-t)} (S_T - k)^+ \,\middle|\, \mathcal{F}_t \right) = v(t, S_t)$$

$$= S_t N(d_1) - k e^{-r(T-t)} N(d_2)$$

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

$$V(T, S_T) = h(S_T)$$

$$V(t, S_t) = \tilde{E}\left(e^{-r(T-t)} h(S_T) \middle| \mathcal{F}_t\right)$$

$$dS_t = rS(t)\,dt + \sigma S(t)\,d\tilde{W}_t$$

* pay off of T : $1(S_T \geq K)$: digital call option

$$V(t, S_t) = \tilde{E}\left(e^{-r(T-t)} 1(S_T \geq K) \middle| \mathcal{F}_t\right)$$

$$= \int_0^\infty e^{-r(T-t)} 1(y \geq K) \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}} dy$$

$$= \int_K^\infty e^{-r(T-t)} \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}} dy$$

$$z = \frac{\ln y - m}{v} \qquad dy = vz\,dz$$

$$= \int_{\frac{\ln K - m}{v}}^\infty e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= e^{-r(T-t)} N\left(\frac{m - \ln k}{v}\right)$$

$$m = \ln S_t + \left(r - \frac{\sigma^2}{2}\right)(T-t)$$

$$v = \sigma\sqrt{T-t}$$

$$= e^{-r(T-t)} N(d_2)$$

price an option which pays $(S_T^2 - k)^+$ at Time T

$$y_t = S_t^2, \quad dS_t = rS_t\,dt + \sigma S_t\,d\tilde{W}_t$$

pay off of the security would be $(y_T - k)^+$

$$V(t, S_t) = \tilde{E}\left((y_T - k)^+ \mid \mathcal{F}_t\right) e^{-r(T-t)}$$

~~$\ln y_t = 2\ln S_t$~~

$$dy_t = 2S_t\,dS_t + \frac{1}{2} \times 2\,(dS_t)^2$$

$$= 2S_t\left(rS_t\,dt + \sigma S_t\,d\tilde{W}_t\right) + \sigma^2 S_t^2\,dt$$

$$= S_t^2(2r + \sigma^2)\,dt + 2\sigma S_t^2\,d\tilde{W}_t$$

$$dy_t = (2r + \sigma^2)\,y_t\,dt + 2\sigma y_t\,d\tilde{W}_t$$

Compute the density of y_T conditioned on $\mathcal{F}_t$ (Derive this)

$$\ln y_T \sim N(m, v^2)$$

$$m = \ln y_t + (2r + \sigma^2 - 2\sigma^2)(T-t)$$

$$v^2 = 4\sigma^2 (T-t)$$

$$P(y_T = y) = \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}}$$

$$V(t, S_t) = \int_0^\infty (y-K)^+ e^{-r(T-t)} \frac{1}{\sqrt{2\pi v^2}\, y} e^{-\frac{(\ln y - m)^2}{2v^2}}\, dy$$

Exercise: Compute this in closed form

Generalised B·S formula

$$V(t, S_t) = \tilde{E}\left(e^{-r(T-t)} (y_T - K)^+ \mid \mathcal{F}_t\right)$$

$$\cancel{dy_t =}\ dS_t = rS_t\, dt + \sigma S_t\, d\tilde{W}_t$$

$y_t = f(S_t)$ f is s.t

y_t is G·B·M

$$dy_t = a y_t\, dt + b y_t\, d\tilde{W}_t \quad (b > 0)$$

Derive this using the transition density

$$y_t e^{a(T-t)} e^{-r(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)$$

$$d_1 = \frac{\ln \frac{y_t}{k} + \left(a + \frac{b^2}{2}\right)(T-t)}{b\sqrt{T-t}}$$

$$d_2 = \frac{\ln \frac{y_t}{k} + \left(a - \frac{b^2}{2}\right)(T-t)}{b\sqrt{T-t}}$$

$$v(t, S_t) = \tilde{E}\left(e^{-r(T-t)} S_T^{\beta} \mid \mathcal{F}_t\right)$$

$$dS_t = r S_t dt + \sigma S_t d\tilde{W}_t$$

$$= \tilde{E}\left(e^{-r(T-t)} (S_T^{\beta} - 0)^+ \mid \mathcal{F}_t\right)$$

$$y_t = S_t^{\beta}$$

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payoff of the option at time T is $\left(\frac{1}{S_T} - k\right)^+$

$$v(t, S_t) = \tilde{E}\left(e^{-r(T-t)} \left(\frac{1}{S_T} - k\right)^+ \mid \mathcal{F}_t\right)$$

$$dS_t = r S(t) dt + \sigma S(t) d\tilde{W}_t$$

$$y_t = \frac{1}{S_t}$$

$$\tilde{E}\left(e^{-r(T-t)} (y_T - k)^+ \mid \mathcal{F}_t\right)$$

$$dy_t = -\frac{1}{S_t^2}\,dS_t + \frac{1}{2}\frac{2}{S_t^3}(dS_t)^2$$

$$= \frac{-1}{S_t^2}\left\{ rS(t)\,dt + \sigma S(t)\,d\tilde{W}_t \right\} + \frac{1}{S_t^3}\sigma^2 S^2(t)\,dt$$

$$= -\frac{1}{S_t}\left\{ r\,dt + \sigma\,d\tilde{W}_t \right\} + \frac{\sigma^2}{S_t}\,dt$$

$$= -y_t\left\{ r\,dt + \sigma\,d\tilde{W}_t \right\} + \sigma^2 y_t\,dt$$

$$dy_t = (\sigma^2 - r)\,y_t\,dt - \sigma\,y(t)\,d\tilde{W}_t \qquad a = \sigma^2 - r,\quad b = \sigma$$

$$d\ln y_t = \left(\sigma^2 - r - \frac{\sigma^2}{2}\right)dt - \sigma\,d\tilde{W}_t \quad \text{(Ito's formula)}$$

$$\ln y_T = \ln y_t + \left(\frac{\sigma^2}{2} - r\right)dt - \sigma\left(\tilde{W}_T - \tilde{W}_t\right)$$

$$\ln y_T \sim N(m, v^2)$$

$$m = \ln y_t + \left(\frac{\sigma^2}{2} - r\right)dt$$

$$v^2 = \sigma^2(T-t) \qquad v = \sigma\sqrt{T-t}$$

$$\tilde{E}\left(e^{-r(T-t)}(y_T - K)^+ \mid \mathcal{F}_t\right)$$

$$= y_t\,e^{-r(T-t)}\,e^{a(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$= \frac{1}{S_t}\,e^{-r(T-t)}\,e^{a(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_1 = \frac{\ln \frac{y_t}{k} + \left(a + \frac{b^2}{2}\right)(T-t)}{b\sqrt{T-t}}$$

$$d_2 = \frac{\ln \frac{y_t}{k} + \left(a - \frac{b^2}{2}\right)(T-t)}{b\sqrt{T-t}}$$

$$y_t = \frac{1}{S_t}$$