(Tower)

① $= E\left(E(X|\mathcal{G}) \,\middle|\, \{\phi, \Omega\} \right) = E\left(X \,\middle|\, \{\phi, \Omega\} \right) = \underline{EX}$

$\quad \| $

$E\,E(X|\mathcal{G})$

② $\boxed{E\left(E(X|\mathcal{G})\right) = \int_\Omega E(X|\mathcal{G})\,dP = \int_\Omega X\,dP = \underline{E}X}$

$E(X|\mathcal{G})$ is a R.V. ① $\mathcal{G}$ - meas

② (Partial Av): $\forall A \in \mathcal{G}$, $\int_A E(X|\mathcal{G})\,dP$

$= \int_A X\,dP.$

Problems.

Q1: $0 \le r < s < t$: Compute $E\big(W(r)W(s)W(t)\big)$.

Note: $E\,W(s)W(t) = s \wedge t = s$.

Sol: $E\big(W(r)W(s)W(t)\big) = E\Big(E\big(W(r)W(s)W(t) \mid \mathcal{F}_s\big)\Big)$.

$= E\Big(W(r)W(s)\underbrace{E\big(W(t) \mid \mathcal{F}_s\big)}_{W(s)}\Big)$. ($\because$ $W(r), W(s)$ are $\mathcal{F}_s$ meas.)

$= E\big(W(r)W(s)^2\big)$.

$$= E\left(E\left(W(r)\,W(s)^2 \,\middle|\, \mathcal{F}_r\right)\right).$$

$$= E\left(W(r)\,E\left(W(s)^2 \,\middle|\, \mathcal{F}_r\right)\right) \ldots \quad * .$$

Compute $E\left(W(s)^2 \,\middle|\, \mathcal{F}_r\right)$:

OPTION 1: $E\left(W(s)^2 \,\middle|\, \mathcal{F}_r\right) = E\left(\left(W(s)-W(r)+W(r)\right)^2 \,\middle|\, \mathcal{F}_r\right)$

$$= E\left(\left(W(s)-W(r)\right)^2 + W(r)^2 + 2\left(W(s)-W(r)\right)W(r) \,\middle|\, \mathcal{F}_r\right).$$

$$= s-r+W(r)^2 + 0\cdot W(r) = (s-r)+W(r)^2.$$

OPTION 2: $\boxed{\text{Itô Formula}}$

$$f(s,x) = x^2.$$

$$W(s)^2 = f(X(s)) \quad , \quad \text{where } \cancel{f(x) = x^2}.$$

$$d\left(W(s)^2\right) = \partial_s f(s, W(s))\, ds + \partial_x f(s, W(s))\, dW(s) + \frac{1}{2}\partial_x^2 f(s, W(s))\, d[W,W](s).$$

$$= 0 + 2W(s)\, dW(s) + ds.$$

$$\Rightarrow W(s)^2 - W(0)^2 = \int_0^s 2W(u)\, dW(u) + \int_0^s du$$

$$\Rightarrow W(s)^2 - W(r)^2 = \int_r^s 2W(u)\, dW(u) + \int_r^s du$$

$$\Rightarrow W(s)^2 = W(r)^2 + \int_{r}^{s} 2W(u)\,dW(u) + \overset{s-r}{\cancel{(r-s)}}.$$

$$\Rightarrow E\left(W(s)^2 \,\middle|\, \mathcal{F}_r\right) = E\left(W(r)^2 + \int_{r}^{s} 2W(u)\,dW(u) + s - r \,\middle|\, \mathcal{F}_r \right)$$

$$= W(r)^2 + \underbrace{\int_{r}^{r} 2W(u)\,dW(u)}_{0} + s - r$$

$$\Rightarrow E\left(W(s)^2 \,\middle|\, \mathcal{F}_r\right) = W(r)^2 + s - r.$$

$$M(t) = \int_0^t \sigma(u)\,dW(u) \Rightarrow M \text{ is a mg} \Rightarrow E\left(M(t) \,\middle|\, \mathcal{F}_s\right) = M(s).$$

$$\Rightarrow E\left(\int_0^t \sigma(u)\,dW(u) \,\middle|\, \mathcal{F}_s\right) = \int_0^s \sigma(u)\,dW(u)$$

$\therefore$ From $\circledast$: $E\big(W(r)W(s)W(t)\big) = E\big(W(r)\, E(W(s)^2 \mid \mathcal{F}_r)\big)$.

$$= E\big(W(r)(W(r)^2 + (s-r))\big) = 0$$

Q2: $X(t) = \cancel{\int_0^{W(t)} e^{-W(s)^2}\, ds}$ $\quad \int_0^{W(t)} e^{-s^2}\, ds.$

~~Compute~~ Write X as $M + B$

$M \to$ Mg

$B \leftarrow$ Finite first Var.

Want $X(t) = \int_0^t \sigma(s)\, dW(s) + \int_0^t b(s)\, ds.$

σ, b adapted.

<u>Ito Formula:</u> $X(t) = f(t, W(t))$ where.

$$f(t, x) = \int_0^x e^{-y^2}\, dy = \int_0^x e^{-s^2}\, ds.$$

Itô $dX = \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2} \partial_x^2 f \, d[W,W](t)$.

Compute: $\partial_t f = 0$

$\underline{\partial_x f = e^{-x^2}}$ $\quad \partial_x^2 f = e^{-x^2}(-2x) = -2x e^{-x^2}$.

$$\Rightarrow dX(t) = 0 + e^{-W(t)^2} dW(t) + \frac{1}{2}\left(-2W(t) e^{-W(t)^2}\right) dt$$

$$\Rightarrow X(t) - X(0) = \int_0^t e^{-W(s)^2} dW(s) \cancel{+} - \int_0^t W(s) e^{-W(s)^2} ds.$$

Compute $[X,X]$: $[X,X](t) = \int_0^t e^{-2W(s)^2} ds$ $\quad \boxed{(dW)^2 = ds}$

$$Y(t) = \exp\left(\int_0^t W(s)\, ds \right) = e^{\int_0^t W(s)\, ds}.$$

Compute $[Y,Y](t)$:

Process: $\boxed{X(t) = \int_0^t W(s)\, ds}$.

$d(f(X))$

Function. $f(t,x) = e^{x}$.

$$\partial_t f = 0, \quad \partial_x f = e^{x} \quad \partial_x^2 f = e^{x}$$

$$d[X,X] = 0$$

$$dY(t) = \left(e^{\int_0^t W(s)\, ds} \right) \cancel{dt}\, dX + 0 \; dW$$

$$\rightarrow [Y,Y](t) = 0$$

$$dY = \cancel{\partial}\left(e^{\int_0^t W(s)\,ds}\right) \underbrace{dX} + 0\,dW + 0\,dt.$$

$$dX = W(t)\,dt.$$

$$dY(t) = W(t)\,e^{\int_0^t W(s)\,ds} \underbrace{dt} + 0\,dW.$$

NOT dW.

Q 3: $X(t) = \int_0^t W(s)\,ds$ & $Y(t) = \int_0^t W(s)\,dW(s)$.

Compute $E\left(X(t) \mid \mathcal{F}_s\right)$ & $E\left(Y(t) \mid \mathcal{F}_s\right)$. $(s < t)$.

① $X(t) = \int_0^t W(r)\,dr$. $\quad E\left(X(t) \mid \mathcal{F}_s\right)$.

$$E\left(\int_0^t W(r)\,dr \,\Big|\, \mathcal{F}_s\right) = \int_0^t E\left(W(r) \mid \mathcal{F}_s\right)\,dr$$

$$\boxed{\underline{\underline{\text{NOT}}} \int_0^t W(s)\,dr}$$

$$= \int_0^s \underbrace{E\left(W(r) \mid \mathcal{F}_s\right)}_{r \leq s}\,dr + \int_s^t \underbrace{E\left(W(r) \mid \mathcal{F}_s\right)}_{r \geq s,\ W \text{ a mg}}\,dr$$

$$= \int_0^s W(r)\,dr + \int_s^t W(s)\,dr = \int_0^s W(r)\,dr + (t-s)\,W(s)$$

② $Y(t) = \int_0^t W(\tau)\, dW(\tau)$.

$$E\left(Y(t) \mid \mathcal{F}_s\right) = \int_0^s W(\tau)\, dW(\tau)$$
$$= Y(s).$$

Itô integrals are Mg's.

Q: $\sigma = \sigma(t)$ Non random (determistic).

Let $X(t) = \int_0^t \sigma(u)\, dW(u)$.

Step 1: $\lambda \in \mathbb{R}$, $0 \le s < t$. $\boxed{\text{Compute } E\left(e^{\lambda(X(t)-X(s))} \middle| \mathcal{F}_s\right)}$.

Ito: Use Itô on $e^{\lambda X(t)}$. : Process is X

Fn $f(x) = e^{\lambda x}$.

$$\partial_t f = 0, \quad \partial_x f = \lambda e^{\lambda x}, \quad \partial_x^2 f = \lambda^2 e^{\lambda x}.$$

$$d\left(e^{\lambda X(t)}\right) = 0\,dt + \lambda e^{\lambda X(t)} dX(t) + \frac{1}{2} \lambda^2 e^{\lambda X(t)} d[X,X](t).$$

$$\Rightarrow d\left(e^{\lambda X(t)}\right) = \lambda e^{\lambda X(t)} \sigma(t)\, dW(t) + \frac{\lambda^2}{2} e^{\lambda X(t)} \sigma^2(t)\, dt$$

$$e^{\lambda X(t)} - e^{\lambda X(s)} = \int_s^t \lambda e^{\lambda X(r)} \sigma(r)\, dW(r) + \frac{\lambda^2}{2} \int_s^t e^{\lambda X(r)} \sigma^2(r)\, dr.$$

$$\Rightarrow e^{\lambda(X(t)-X(s))} - 1 = e^{-\lambda X(s)} \int_s^t \lambda e^{\lambda X(r)} \sigma(r)\, dW(r)$$

$$+ \frac{\lambda^2}{2} \int_s^t e^{\lambda(X(r)-X(s))} \sigma^2(r)\, dr.$$

$$\boxed{\text{Let } \varphi(t) = E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right).}$$

$$\Rightarrow \varphi(t) - 1 = E\left(e^{-\lambda X(s)} \int_s^t \lambda e^{\lambda X(r)} \sigma(r)\, dW(r) \,\Big|\, \mathcal{F}_s \right)$$

$$+ \frac{\lambda^2}{2} \int_s^t \sigma(r)^2 \varphi(r)\, dr$$

Note: $E\left(e^{-\lambda X(s)} \int_s^t (\,)\, dW(r) \,\Big|\, \mathcal{F}_s \right) = e^{-\lambda X(s)} \int_s^s (\,)\, dW = 0$

$$\Rightarrow \varphi(t) = 1 + 0 + \frac{\lambda^2}{2} \int_s^t \sigma(r)^2 \varphi(r)\, dr$$

$$\Rightarrow \frac{d\varphi}{dt} = \frac{\lambda^2}{2} \sigma(t)^2 \varphi(t) \Rightarrow \varphi(t) = 1\, e^{\frac{\lambda^2}{2} \int_s^t \sigma(r)^2\, dr}$$

$$\Rightarrow E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right) = e^{\frac{\lambda^2}{2}\int_s^t \sigma(r)^2\,dr} \quad \cdots \quad (**)$$

$$\boxed{\begin{aligned} &\Rightarrow \text{MGF of } X(t)-X(s) = e^{\frac{\lambda^2}{2}\int_s^t \sigma(r)^2\,dr} \\ &\Rightarrow X(t)-X(s) \sim N\left(0, \int_s^t \sigma(r)^2\,dr\right). \end{aligned}}$$

② Compute Joint MGF $E\left(e^{\mu X(r) + \lambda(X(t)-X(s))}\right)$ for $r \le s$.

$$E\left(e^{\mu X(r) + \lambda(X(t)-X(s))}\right) = E\,E\left(e^{\mu X(r)} e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right)$$

$$= E\, e^{\mu X(r)} \underbrace{E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right)}.$$

$$= \underbrace{E\, e^{\mu X(r)}} \cdot \underbrace{e^{\frac{\lambda^2}{2}\int_s^t \sigma(u)^2\,du}}_{\text{not random}}.$$

$$= e^{\frac{\mu^2}{2}\int_0^r \sigma(u)^2\,du} \cdot e^{\frac{\lambda^2}{2}\int_s^t \sigma(u)^2\,du}.$$

$$= MGF_{X(r)}(\mu) \cdot MGF_{X(t)-X(s)}(\lambda).$$

$\Rightarrow$ $X(t)-X(s)$ is ind of $X(r)$

& $(X(r), X(t)-X(s))$ is Gaussian. $\begin{pmatrix} \int_0^r \sigma(u)^2\,du & 0 \\ 0 & \int_s^t \sigma(u)^2\,du \end{pmatrix}$.