Itô's formula

Itô process $$X_t = X_0 + \overbrace{\int_0^t \mu_s\,ds + \int_0^t \sigma_s}^{\text{adapted process.}}\,dW_s$$

$$dX_t = \mu_t\,dt + \sigma_t\,dW_t$$

$$[X,X]_t = \int_0^t \sigma_s^2\,ds = \int_0^t \sigma_s^2\,d[W,W]_s$$

Itô's formula $\quad f \in C^{1,2} \qquad f = f(\underset{\text{time}}{t}, \underset{\text{spatial}}{x})$

$$f(t,X_t) = f(0,X_0) + \int_0^t f_t(s,X_s)\,ds + \int_0^t f_x(s,X_s)\,dX_s + \frac{1}{2}\int_0^t f_{xx}(s,X_s)\,d[X,X]_s$$

$$= f(0,X_0) + \int_0^t f_t(s,X_s)\,ds + \left(\int_0^t f_x(s,X_s)\,\mu_s\,ds + \int_0^t f_x(s,X_s)\,\sigma_s\,dW_s\right)$$
$$+ \frac{1}{2}\int_0^t f_{xx}(s,X_s)\cdot\sigma_s^2\,ds.$$

<u>Ex 1</u> $S \sim GBM(\mu,\sigma^2)$ if $S_t = \underline{S_0} e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma \underline{W_t}}$

Compute dS_t

Apply Itô to $f(t,x) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma x}$.

$$f_t(t,x) = (\mu - \tfrac{1}{2}\sigma^2)\,f(t,x)$$
$$f_x(t,x) = \sigma f(t,x), \quad f_{xx}(t,x) = \sigma^2 f(t,x).$$

$$S_t = f(t, W_t)$$

$$dS_t = df(t, W_t) = f_t(t, W_t)\,dt + f_x(t, W_t)\,dW_t + \frac{1}{2} f_{xx}(t, W_t)\,dt$$

$$= (\mu - \frac{1}{2}\sigma^2) S_t\,dt + \sigma S_t\,dW_t + \frac{1}{2}\sigma^2 S_t\,dt$$

$$= \underbrace{S_t(\mu\,dt + \sigma\,dW_t)} \qquad S_t \sigma\,dW_t$$

(a) dS_t^p, $S_t^p \sim GBM(\alpha, \theta^2)$, $f(x) = x^p$

$$d(S_t^p) = pS_t^{p-1}\,dS_t + \frac{1}{2}p(p-1)S_t^{p-2}\,d[S,S]_t$$

$$= pS_t^{p-1} S_t(\mu\,dt + \sigma\,dW_t) + \frac{1}{2}p(p-1)S_t^{p-2} S_t^2 \sigma^2\,dt$$

$$= S_t^p(p\mu + \frac{1}{2}\sigma^2 p(p-1))\,dt + p\sigma S_t^p\,dW_t$$

$$= S_t^p\left(\underbrace{[p\mu + \frac{1}{2}\sigma^2 p(p-1)]\,dt} + \underbrace{p\sigma\,dW_t}_{\text{martingale}}\right)$$

$$\alpha = p\mu + \frac{1}{2}\sigma^2 p(p-1), \quad \theta = p\sigma$$

For what p is S_t^p a martingale?

martingale $\Rightarrow$ dt term $\equiv 0$

$$p\mu + \frac{1}{2}\sigma^2 p(p-1) = 0 \Rightarrow \begin{cases} p = 0 \\ p = 1 - \frac{2\mu}{\sigma^2} \end{cases}$$

Ex2

$$X_t = e^{-\lambda t} \overbrace{\int_0^t \underline{e^{\lambda u} dW_u}}^{Y_t} \quad , \lambda > 0$$

Compute $dX_t = \underbrace{\mu_t dt + \sigma_t dW_t}$

$$X_t = e^{-\lambda t} Y_t = f(t, Y_t) \quad , \quad f(t,y) = e^{-\lambda t} y$$

$$f_t = -\lambda e^{-\lambda t} y = -\lambda f$$

$$f_y = e^{-\lambda t} \quad , \quad f_{yy} = 0$$

Itô $\Rightarrow$

$$dX_t = d f(t, Y_t) = -\lambda f(t, Y_t) dt + e^{-\lambda t} dY_t + 0$$

$$= -\lambda X_t dt + e^{-\lambda t} e^{\lambda t} dW_t$$

$$= -\lambda X_t \, dt + dW_t$$

deterministic. $\sigma^2(X_t)\,dt$

$$dX_t = \cancel{\mu(X_t)\,dt} + \sigma(X_t)\,dW_t \qquad \text{(SDE)}.$$

X_t is called a sol'n to the stochastic differential eqn (SDE).

Consider $f(t, X_t)$, $f = f(t,x)$.

Look for a differential equation that f has to solve, in order to have $f(t, X_t)$ a m'gale.

$$df(t, X_t) = f_t(t, X_t)\,dt + f_x(t, X_t)\,dX_t + \frac{1}{2} f_{xx}(t, X_t)\,d[X, X]_t$$

$$= f_t(t, X_t)\,dt + f_x(t, X_t)\big(\mu(X_t)\,dt + \sigma(X_t)\,dW_t\big)$$

$$+ \frac{1}{2} f_{xx}(t, X_t)\,\sigma^2(X_t)\,dt$$

$$= \Big[\underbrace{f_t(t, X_t) + \mu(X_t) f_x(t, X_t) + \frac{1}{2} f_{xx}(t, X_t)\,\sigma^2(X_t)}_{=0}\Big]\,dt$$

$$+ \quad dW_t$$

$$f_t(t, x) + \mu(x) f_x(t, x) + \frac{1}{2}\sigma^2(x) f_{xx}(t, x) = 0$$

$f : [0,T] \to \mathbb{R}$ deterministic.

Ex3 $\int_0^T f_u \, dW_u$ ← distribution?

$\sim N(0, \int_0^T f_u^2 \, du)$

$$\mathbb{E}\left(e^{t \cdot \int_0^T f_u \, dW_u} \right) = e^{\frac{1}{2} t^2 \int_0^T f_u^2 \, du}$$

$$\mathbb{E}\left(\underbrace{e^{t \int_0^T f_u \, dW_u - \frac{1}{2} t^2 \int_0^T f_u^2 \, du}} \right) = 1$$

$$Z_s = e^{t \underbrace{\int_0^s f_u \, dW_u}_{=0} - \frac{1}{2} t^2 \underbrace{\int_0^s f_u^2 \, du}_{=0}}$$

$Z_0 = 1$

if Z is a martingale, $\mathbb{E}(Z_T) = \mathbb{E}(Z_0) = 1$

$$X_s = \underline{\boxed{t \int_0^s f_u\,dW_u} - \frac{1}{2}t^2\int_0^s f_u^2\,du}$$

$$Z_s = e^{X_s}$$

Itô $\Rightarrow$ $dZ_s = e^{X_s}\,dX_s + \frac{1}{2}e^{X_s}\,d[X,X]_s$

$$= e^{X_s}\left(t\cdot f_s\,dW_s - \underline{\frac{1}{2}t^2 f_s^2\,ds}\right) + \underline{\frac{1}{2}e^{X_s}t^2 f_s^2\,ds}$$

$$= e^{X_s}\,t\,f_s\,dW_s$$

$\Rightarrow$ Z is a m'gale.

$\underline{\mathcal{E}x4}$: $f:[0,T]\to\mathbb{R}$ deterministic.

$$X_T = \int_0^T \boxed{f_s W_s}\, ds \quad \text{distribution???}$$

$g(t,x)$

$$dg(t,W_t) = g_t(t,W_t)\,dt + g_x(t,W_t)\,dW_t + \tfrac{1}{2} g_{xx}(t,W_t)\,dt$$

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$\underline{d(h(t)W_t)}$

$$= \underbrace{\left[g_t(t,W_t) + \tfrac{1}{2} g_{xx}(t,W_t)\right]}_{①} dt + \underbrace{g_x(t,W_t)}_{②}\, dW_t$$

$$= f_t W_t\, dt \quad + \quad -h(t)\, dW_t$$

$$\begin{cases} \text{②} \quad f_t W_t = g_t(t, W_t) + \frac{1}{2} g_{xx}(t, W_t). \\ \text{①} \quad g_x(t, W_t) \leftarrow \text{deterministic.} \end{cases}$$

$$\underline{g(t,x) = h(t)\,x} , \quad g_x(t,x) = h(t)$$

$$g_t(t, W_t) + \frac{1}{2} g_{xx}(t, W_t) = \underline{h'(t)}\, W_t + 0$$

$$h' = f \implies \underline{h(t) = \int_0^t f_u\,du}$$

$$g(t,x) = x \cdot h(t) = x \int_0^t f_u\,du.$$

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + d[X,Y]_t$$

$$\checkmark \quad \int_0^T \alpha f_t + \beta g_t \, dW_t = \alpha \int_0^T f_t \, dW_t + \beta \int_0^T g_t \, dW_t$$

(b) $$dX_t = d(W_t^2) + (W_t^3 - 1)\,dt \qquad f(x) = x^2$$

$$= \left(2W_t\,dW_t + \frac{1}{2}\cdot 2\underbrace{d[W,W]_t}_{dt}\right) + (W_t^3 - 1)\,dt$$

$$= \underline{W_t^3\,dt} + 2W_t\,dW_t$$