# STOCH CAL PROBLEM SOLVING SESSION - 2 

3 $X$ & $Y$ are independent and standard Normal Random variables

$$E\left(e^{XY/2}\right) = E\left(E\left(e^{XY/2} \mid Y = y\right)\right)$$

$$E\left(e^{XY/2} \mid Y=y\right) = E\left(e^{\frac{yX}{2}} \mid Y=y\right) = E\left(e^{yX/2}\right)$$

$$= e^{\frac{y^2}{4 \times 2}}$$

$$= e^{y^2/8}$$

If $X \sim N(\mu, \sigma^2)$

$$E(e^{tX}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$E(f(X) \mid Y) = g(Y)$

$\boxed{\cancel{E(X|G)} = \cancel{h}(G)}$

$$E\left(e^{XY/2} \mid Y=y\right) = e^{y^2/8}$$

$$E\left(e^{XY/2} \mid Y\right) = e^{\frac{y^2}{8}} = e^{\frac{Y^2}{8}}$$

$$E\left(e^{\frac{XY}{2}}\right) = E\left(e^{\frac{Y^2}{8}}\right)$$

$$= \int_{-\infty}^{\infty} e^{y^2/8} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$\frac{1}{2} - \frac{1}{8}$

$\frac{3}{8}$

$$= \int_{-\infty}^{\infty} \sigma \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{3}{8}y^2} dy$$

$2\sigma^2 = \frac{8}{3} \Rightarrow \sigma^2 = \frac{4}{3}$

$\frac{4}{\sigma} = 2$

$$= \frac{2}{\sqrt{3}}$$

2- Compute the distribution of $\int_0^t f(s)\,dW_s$

$f(s)$ is a deterministic function.

$$E\left(e^{\lambda\int_0^t f(s)\,dW_s}\right)$$

~~$e^{M_t - \frac{\langle M\rangle_t}{2}}$ is a~~

if $M_t$ is a Martingale, then $e^{M_t - \frac{\langle M\rangle_t}{2}}$ is a Martingale

$$M_t = \lambda\int_0^t f(s)\,dW_s \qquad \langle M\rangle_t = \lambda^2\int_0^t f^2(s)\,ds$$

$$Z_t = e^{\int_0^t X_s\,dW_s - \frac{1}{2}\int_0^t X_s^2\,ds}$$

$$Y_t = \int_0^t X_s\,dW_s - \frac{1}{2}\int_0^t X_s^2\,ds$$

$$Z_t = e^{Y_t}$$

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digression $\quad Y_t = f(X_t)$

$$dY_t = f'(X_t)\,dX_t + \frac{1}{2} f''(X_t)(dX_t)^2$$

$$Y_t = f(t, X_t)$$

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)\,dt + \frac{\partial f}{\partial x}(t, X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$$

$$Z_t = f(X_t, Y_t)$$

$$dZ_t = \frac{\partial f}{\partial x}\,dX_t + \frac{\partial f}{\partial y}(X_t, Y_t)\,dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(dY_t)^2 + \frac{\partial^2 f}{\partial x\,\partial y}(dX_t)\,dY_t$$

Prove

$$Z_t = e^{\int_0^t X_s\,dW_s - \frac{1}{2}\int_0^t X_s^2\,ds} \quad \text{is a Martingale}$$

$$Y_t = \int_0^t X_s\,dW_s - \frac{1}{2}\int_0^t X_s^2\,ds \qquad dY_t = X_t\,dW_t - \frac{X_t^2}{2}\,dt$$

$$Z_t = e^{Y_t}$$

$$(dW_t)^2 = dt$$

$$(dt)^2 = 0$$

$$dW_t \cdot dt = 0$$

$$dZ_t = e^{Y_t}\,dY_t + \frac{1}{2}e^{Y_t}(dY_t)^2$$

$$dZ_t = e^{Y_t}\left(X_t\,dW_t - \frac{X_t^2\,dt}{2} + \frac{1}{2}X_t^2\,dt\right)$$

$$dZ_t = X_t Z_t\,dW_t \Rightarrow Z_t = \int_0^t X_s Z_s\,dW_s$$

$\Rightarrow$ $Z_t$ is a Martingale.

$$Z_t = e^{M_t - \frac{\langle M\rangle_t}{2}}$$ is a Martingale if $M_t$ is a Martingale

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(2) Compute the distribution of $\int_0^t f(s)\,dW_s$ $f(s)$ is a deterministic function

$$E\left(e^{\lambda\int_0^t f(s)\,dW_s}\right)$$

$$Z_t = e^{\lambda\int_0^t f(s)\,dW_s - \frac{1}{2}\int_0^t \lambda^2 f^2(s)\,ds}$$

$$E(Z_t) = E(Z_0) = 1$$

$$E\left(e^{\lambda\int_0^t f(s)\,dW_s - \frac{1}{2}\int_0^t \lambda^2 f^2(s)\,ds}\right) = 1$$

$$E(g(s)h(Y)) = g(s)\,E(h(Y))$$

$$E\left(e^{\lambda\int_0^t f(s)\,dW_s}\right) = e^{\frac{1}{2}\int_0^t \lambda^2 f^2(s)\,ds} \qquad \sigma^2 = \int_0^t f^2(s)\,ds$$

$$\uparrow$$

$$e^{-\frac{1}{2}\int_0^t \lambda^2 f^2(s)\,ds}\; E\left(e^{\lambda\int_0^t f(s)\,dW_s}\right) = 1$$

$$E\left(e^{\lambda\int_0^t f(s)\,dW_s}\right) = e^{\frac{\lambda^2\sigma^2}{2}}$$

$$\Rightarrow \int_0^t f(s)\,dW_s \sim N\left(0, \int_0^t f^2(s)\,ds\right)$$

$$\int_0^t W_s\,dW_s \not\to \text{Normally distributed}$$

1. $X_t = e^{tW_t}$

Find the differential ~~a~~ $dX_t$ of the process $X_t$

$$X_t = e^{Y_t} \qquad Y_t = tW_t$$

$$dX_t = e^{Y_t}\,dY_t + \frac{1}{2}e^{Y_t}(dY_t)^2$$

$$Y_t = tW_t$$

$$\boxed{{}^{*}\,dX_tY_t = X_t\,dY_t + Y_t\,dX_t + dX_t\,dY_t}$$

$$dY_t = t\,dW_t + W_t\,dt + \underbrace{dt\,dW_t}_{0}$$

$$dY_t = t\,dW_t + W_t\,dt$$

$$dX_t = X_t\left(dY_t + \frac{1}{2}(dY_t)^2\right)$$

$$= X_t\left(t\,dW_t + W_t\,dt + \frac{1}{2}t^2\,dt\right)$$

$$dX_t = X_t\left(W_t + \frac{t^2}{2}\right)dt + tX_t\,dW_t$$

Compute $E(e^{\alpha W(t)})$ $\qquad \langle Y_t \rangle = \int_0^t (dY_s)^2$

$$Z_t = e^{\alpha W_t - \frac{\alpha^2 t}{2}} \Rightarrow Z_t \text{ is a Mg} \qquad W(t) = \sqrt{t}\, Z$$

$$E(Z_t) = E(Z_0) = 1$$

$$E\left(e^{\alpha W_t - \frac{\alpha^2 t}{2}}\right) = 1$$

$$E\left(e^{\alpha W_t}\right) = e^{\frac{\alpha^2 t}{2}}$$

$$E\left(e^{\alpha \sqrt{t} Z}\right) = e^{\frac{\alpha^2 t}{2}}$$

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$$\int_0^t W_s \, dW_s$$

$$dW_t^2 = 2 W_t \, dW_t + \frac{1}{2} \times 2 \, dt$$

$$\Rightarrow W_t^2 = 2 \int_0^t W_s \, dW_s + t$$

$$\Rightarrow \frac{W_t^2 - t}{2} = \int_0^t W_s \, dW_s$$

$$\int_0^t W_s^{n-1} \, dW_s$$

$$dW_t^n = n W_t^{n-1} \, dW_t + \frac{1}{2} n(n-1) W_t^{n-2} \, dt$$

$$W_t^n - \frac{1}{2} \int_0^t n(n-1) W_s^{n-2} \, ds = n \int_0^t W_s^{n-1} \, dW_s$$

1. Find the distribution of the Random variable

$$\int_0^t W_s\,ds$$

$$d(tW_t) = t\,dW_t + W_t\,dt + \underbrace{dt\,dW_t}_{0}$$

$$tW_t - \int_0^t s\,dW_s = \int_0^t W_s\,ds$$

$$df(t,X_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX_t)^2$$

$$f(t,W_t) = tW_t$$

$$df(t,W_t) = W_t\,dt + t\,dW_t + \frac{1}{2}\times 0\times (dW_t)^2$$

$$d\,tW_t = W_t\,dt + t\,dW_t$$

$$tW_t = t\int_0^t dW_s$$

$$tW_t - \int_0^t s\,dW_s = t\int_0^t dW_s - \int_0^t s\,dW_s$$

$$= \int_0^t (t-s)\,dW_s = \int_0^t W_s\,ds$$

$$\int_0^t W_s\,ds = \int_0^t (t-s)\,dW_s \sim N\left(0, \int_0^t (t-s)^2\,ds\right)$$

$$\sim N\left(0, \frac{t^3}{3}\right)$$

Let $f(x)$ be a function and $\lambda$ be a number

s.t $f(0) = 1$, $f''(x) = \lambda f(x)$

Prove $E(f(W_t)) = e^{\frac{\lambda t}{2}}$

$$df(W_t) = f'(W_t)\,dW_t + \frac{1}{2} f''(W_t)\,dt$$

$$f(W_t) - f(W_0) = \int_0^t f'(W_s)\,dW_s + \frac{1}{2}\int_0^t f''(W_s)\,ds$$

$$f(W_t) - f(0) = \int_0^t f'(W_s)\,dW_s + \frac{1}{2}\int_0^t f''(W_s)\,ds$$

$$E(f(W_t)) = 1 + \frac{1}{2} E \int_0^t f''(W_s)\,ds$$

$$= 1 + \frac{1}{2} E \int_0^t \lambda f(W_s)\,ds$$

$$E(f(W_t)) = 1 + \frac{1}{2}\lambda \int_0^t E(f(W_s))\,ds$$

$$E(f(W_t)) = y(t)$$

$$y(t) = 1 + \frac{\lambda}{2}\int_0^t y(s)\,ds$$

$$\frac{dy(t)}{dt} = y'(t) = \frac{\lambda}{2} y(t) \qquad \frac{dy(t)}{y(t)} = \frac{\lambda}{2}\,dt$$

$$\ln y(t) = \frac{\lambda}{2} t + c$$

$$\Rightarrow y(t) = K e^{\frac{\lambda t}{2}} = E(f(W_t))$$

$$E(f(W_t)) = ke^{\frac{\lambda t}{2}}$$

Plug $t=0$

$$E(f(W_0)) = K$$

$$\Rightarrow K = 1$$

$$\Rightarrow E(f(W_t)) = e^{\frac{\lambda t}{2}}$$

(4) True or False questions

a) If $W$ & $B$ are independent BMs, then the average of $W$ & $B$ given by $X_t = \frac{1}{2}(W_t + B_t)$ is a again a Brownian Motion

A process $X_t$ is said to be a BM if it satisfies the following properties

(i) $X_0 = 0$

(ii) $E(X_t X_s) = \min(t,s)$ $\quad (E((X_t - X_s + X_s) X_s)$ if $t \geq s$

$\quad = E[(X_t - X_s) X_s] + E(X_s^2)$

$\quad = E(X_t - X_s) E(X_s) = 0$, $\quad E(X_s^2) = s$

(iii) $X_t$ is Gaussian distributed with mean $0$

(iv) $X_t$ has continuous sample paths

Show that $X_t = \frac{W_{ct}}{\sqrt{c}}$ is BM $\quad c > 0$ is a constant

$E(X_t X_s)$ for $t \geq s$

$E\left(\frac{1}{2}(W_t + B_t)\frac{1}{2}(W_s + B_s)\right)$

$\frac{1}{4} E(W_t W_s + \underbrace{W_t B_s}_{0} + \underbrace{B_t W_s}_{0} + B_t B_s)$

$\frac{1}{4}\{s + s\} = \frac{s}{2} \neq s$

Hence $X_t$ is not BM

(b). If $X$ & $Y$ are Martingales then the average of $X$ & $Y$ given by $Z_t = \frac{X_t + Y_t}{2}$ is a Martingale

True : $E(Z_t | \mathcal{F}_s) = Z_s$

(c) $X$ has finite non zero Quadratic variation then $X$ has infinite first variation

True

$$\sum |X_{t_{i+1}} - X_{t_i}|^2 \leq \max_i |X_{t_{i+1}} - X_{t_i}| \sum |X_{t_{i+1}} - X_{t_i}|$$

If the process has finite first variation then $\sum |X_{t_{i+1}} - X_{t_i}| < \infty$

$$\Rightarrow \sum |X_{t_{i+1}} - X_{t_i}|^2 = 0$$

$\cancel{P(X \geq t, N^2 \geq y) = P(X \geq t)\, P(N \geq y)}$

$\cancel{E(f(X)\, g(N)) = \qquad g:}$

$\cancel{E(f(X)\, h(N^2)) =}$

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1 $P(X \leq tY) = \int_{0}^{\infty} P(X \leq tY \mid Y = y)\, P(Y = y)\, dy$

$X \,\&\, Y > 0$

and $X \,\&\, Y \perp\!\!\!\perp$

$X \sim \exp(1)$

$$= \int_0^\infty (1 - e^{-ty})\, f_Y(y)\, dy$$

$Y \sim N^2$

$$= 1 - E(e^{-tY})$$

$$E(e^{-tY}) = E(e^{-tN^2}) = \int_{-\infty}^{\infty} e^{-ty^2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

$$= \frac{1}{\sqrt{1+2t}}$$

$P(Z \leq t) \quad Z = \frac{X}{Y} \quad Y = N^2(0,1)$

(a) $X_t = e^{-t} W_t^2$ $\qquad f(t,x) = e^{-t} x^2$

Obtain the ito process decomposition of $X_t$

$Y_t \; f(t, X_t) \qquad Y_t = f(t, W_t)$

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)\, dt + \frac{\partial f}{\partial x}(t, X_t)\, dX_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$$

$$dX_t = -e^{-t} W_t^2\, dt + e^{-t} 2 W_t\, dW_t + \frac{1}{2} e^{-t} 2\, dt$$

$$dX_t = e^{-t}(1 - W_t^2)\, dt + 2e^{-t} W_t\, dW_t$$

(b) Compute the ito process decomposition of

$$Y_t = \sin X_t$$

$$dY_t = \cos X_t\, dX_t \mp \frac{1}{2} \sin(X_t)(dX_t)^2$$

$$(dX_t)^2 = 4e^{-2t} W_t^2\, dt$$

What is the quadratic variation of $X_t$ ?

Compute $dX_t$

$$\langle X \rangle_t = \int_0^t (dX_s)^2 = \int_0^t 4e^{-2s} W_s^2\, ds$$

(13)

(3) $\{M_t\}$ is a continuous Martingale. Fix a time $t_0 > 0$ and let $\Delta = \{\Delta_t\}_{t \geq 0}$ be the simple process defined by

$$\Delta_t = \begin{cases} \Delta_0 & \text{if } t \leq t_0 \\ \Delta_{t_0} & \text{if } t > t_0 \end{cases}$$

$\Delta_0$ is a constant

$\Delta_{t_0}$ is $\mathcal{F}_{t_0}$ measurable

Define the Stochastic integral of $\Delta$ with respect to $M$ by the formula

$$I_t = \Delta_0 M_{t \wedge t_0} + \Delta_{t_0}(M_t - M_{t \wedge t_0}) \quad t \geq 0$$

$$a \wedge b = \min\{a, b\}$$

(a) Is $I$ a Martingale

$$E(I(t) \mid \mathcal{F}_s) \overset{?}{=} I(s) \qquad t \geq s$$

(i) $s \leq t \leq t_0$

$$I_t = \Delta_0 M_t + \Delta_{t_0}(M_t - M_t)$$

$$I_t = \Delta_0 M_t$$

$$E(I_t \mid \mathcal{F}_s) = E(\Delta_0 M_t \mid \mathcal{F}_s) = \Delta_0 M_s = I_s$$

(ii) $s \leq t_0 \leq t$ $\qquad \mathcal{F}_s \subseteq \mathcal{F}_{t_0}$

$$I(t) = \Delta_0 M_{t_0} + \Delta_{t_0}(M_t - M_{t_0})$$

$$E(I(t)\,|\,\mathcal{F}_s) = E(\Delta_0 M_{t_0} + \Delta_{t_0}(M_t - M_{t_0})\,|\,\mathcal{F}_s)$$

$$= \Delta_0 M_s + E(\Delta_{t_0}(M_t - M_{t_0})\,|\,\mathcal{F}_s)$$

$$= \Delta_0 M_s + E(E(\Delta_{t_0}(M_t - M_{t_0})\,|\,\mathcal{F}_{t_0})\,|\,\mathcal{F}_s)$$

$$E(\Delta_{t_0}(M_t - M_{t_0})\,|\,\mathcal{F}_{t_0}) = \Delta_{t_0} E(M_t - M_{t_0}\,|\,\mathcal{F}_{t_0})$$

$$= 0$$

$$E(I(t)\,|\,\mathcal{F}_s) = \Delta_0 M_s = I(s)$$

$$I(s) = \Delta_0 M_s + \Delta_{t_0}(M_s - M_s) = \Delta_0 M_s$$

(iii) $t_0 \le s \le t$

$$I(t) = \Delta_0 M_{t_0} + \Delta_{t_0}(M_t - M_{t_0})$$

$$E(I(t)\,|\,\mathcal{F}_s) = E(\Delta_0 M_{t_0} + \Delta_{t_0}(M_t - M_{t_0})\,|\,\mathcal{F}_s)$$

$$= \Delta_0 M_{t_0} + \Delta_{t_0}(M_s - M_{t_0})$$

$$= I_s$$

Hence $I(t)$ is a Martingale.