Stochastic Calculus I TA #2

1° Brownian Motion

<u>Ex 1</u>: $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ given.

W, B independent B.M.s

Define $X_t = \rho W_t + \sqrt{1-\rho^2}\, B_t \leftarrow \boxed{\text{B.M.}}$

$\rho \in (-1,1) \setminus \{0\}$.

(1) $X_0 = 0$ a.s. ✓

(2) $t \mapsto X_t = \underline{\rho W_t + \sqrt{1-\rho^2}\, B_t}$ continuous ✓

(3) For $0 \leq s < t$, $X_t - X_s = \rho \underbrace{(W_t - W_s)}_{\sim N(0, \rho^2(t-s))} + \sqrt{1-\rho^2}\, \underbrace{(B_t - B_s)}_{\sim N(0, (1-\rho^2)(t-s))} \sim N(0, t-s)$ ✓

(4) For $0 \leq s < t < u < v$, $X_t - X_s \perp\!\!\!\perp X_v - X_u$

Recall: $\perp\!\!\!\perp \iff \mathrm{Cov} = 0$

- Show $(X_t - X_s, X_v - X_u)$ is jointly normal.

$$\begin{pmatrix} X_t - X_s \\ X_v - X_u \end{pmatrix} = \begin{pmatrix} \rho(W_t - W_s) + \sqrt{1-\rho^2}\,(B_t - B_s) \\ \rho(W_v - W_u) + \sqrt{1-\rho^2}\,(B_v - B_u) \end{pmatrix}$$

$$= \begin{pmatrix} \rho & \sqrt{1-\rho^2} & 0 & 0 \\ 0 & 0 & \rho & \sqrt{1-\rho^2} \end{pmatrix} \boxed{\begin{pmatrix} W_t - W_s \\ B_t - B_s \\ W_v - W_u \\ B_v - B_u \end{pmatrix}}$$

Vector of indep normal.
$\Downarrow$
jointly normal.

$\Rightarrow \begin{pmatrix} X_t - X_s \\ X_v - X_u \end{pmatrix}$ jointly normal

- $$\mathrm{Cov}(X_t - X_s, X_v - X_u) = \mathbb{E}\big((X_t - X_s)(X_v - X_u)\big)$$

$$= \mathbb{E}\Big(\big(\rho\underbrace{(W_t - W_s)}_{1} + \sqrt{1-\rho^2}\underbrace{(B_t - B_s)}_{2}\big)\big(\rho\underbrace{(W_v - W_u)}_{3} + \sqrt{1-\rho^2}\underbrace{(B_v - B_u)}_{4}\big)\Big)$$

$$\stackrel{?}{=} 0$$

$$\underline{Y_t = X_t W_t}$$

$$\mathbb{E}(Y_t) = \mathbb{E}\left(\left(\rho W_t + \sqrt{1-\rho^2}\, B_t\right) W_t \right) = \mathbb{E}(\rho W_t^2) = \boxed{\rho t}$$

NOT a m'gale!

$$= \sqrt{1-\rho^2}\, \underbrace{\mathbb{E}(B_t)}_{=0}\, \mathbb{E}(W_t)$$

2° Conditional Expectation & Martingale

Recall: $Y = \mathbb{E}(X \mid \mathcal{G})$ is a r.v. satisfying

① Y is $\mathcal{G}$-measurable

② (Partial Average) $A \in \mathcal{G}$,

$$\mathbb{E}(Y \mathbb{1}_A) = \mathbb{E}(X \mathbb{1}_A)$$

$$\mathbb{E}(X \mid \underset{\substack{\uparrow \\ \text{info}}}{\mathcal{G}}) \longrightarrow X$$

Properties:

(1) if X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$

(2) if X is independent of $\mathcal{G}$, $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.

(3) linearity, positivity ...

(4) TOWK (taken out what's known)

X is $\mathcal{G}$-measurable, then $\mathbb{E}(XY|\mathcal{G}) = X\,\mathbb{E}(Y|\mathcal{G})$

(5) Tower property, $\mathcal{H} \subseteq \mathcal{G}$, then

$$\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\,|\,\mathcal{H}\big) = \mathbb{E}(X|\mathcal{H})$$

Ex 1: $\mathbb{E}(X|\mathcal{G})$ minimizes mean square error

$(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ given. If Y is $\mathcal{G}$-measurable,

X given $\qquad\qquad$ $\mathbb{E}(Y^2) < \infty$, then

$$\underline{\mathbb{E}(|X-Y|^2)} \geq \mathbb{E}(|X - \mathbb{E}(X|\mathcal{G})|^2) \qquad (*)$$

$$(a+b)^2$$

$$\mathbb{E}\left(|X-Y|^2\right) = \mathbb{E}\left(\Big|\underbrace{X-\mathbb{E}(X|\mathcal{G})}_{a} + \underbrace{\overbrace{\mathbb{E}(X|\mathcal{G})-Y}^{Z}}_{b}\Big|^2\right)$$

$$= \mathbb{E}\left(|X-\mathbb{E}(X|\mathcal{G})|^2\right) + \cancel{2\,\mathbb{E}\left(Z\left(X-\mathbb{E}(X|\mathcal{G})\right)\right)}$$

$$+ \underbrace{\mathbb{E}(Z^2)}_{\geq 0}$$

2nd term $= \mathbb{E}\left(Z\left(X-\mathbb{E}(X|\mathcal{G})\right)\right)$

(tower)

$\mathcal{H}=\{\Omega,\phi\}$

$$= \mathbb{E}\left(\mathbb{E}\left(Z\left(X-\mathbb{E}(X|\mathcal{G})\right)\Big|\,\mathcal{G}\right)\right)$$

TOWK

$$= \mathbb{E}\left(Z\ \mathbb{E}\left([X-\mathbb{E}(X|\mathcal{G})]\,|\,\mathcal{G}\right)\right)$$

$$= \mathbb{E}\left(Z\left(\mathbb{E}(X|\mathcal{G}) - \mathbb{E}(X|\mathcal{G})\right)\right) = 0$$

$=0$

$\boxed{\text{Ex}}$: $\mathrm{Cov}(Y, X - \mathbb{E}(X|\mathcal{G}_1)) = ?$

<u>Ex 2</u>: <u>Recall</u>: $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$

$\mathbb{E}(\mathbb{1}_A | \sigma(\mathbb{1}_B)) \sim$

$\sigma(\mathbb{1}_B) = \{\emptyset, B, B^c, \Omega\}$.

<u>Claim</u>: $\mathbb{E}\left(\mathbb{1}_A \middle| \sigma(\mathbb{1}_B)\right) = \underbrace{\mathbb{P}(A|B)\mathbb{1}_B + \mathbb{P}(A|B^c)\mathbb{1}_{B^c}}_{=X}$.

1 Show X is $\sigma(\mathbb{1}_B) = \mathcal{G}$ measurable.

$$\{X \leq \alpha\} \in \sigma(\mathbb{1}_B)$$

$$X(\omega) = \begin{cases} \mathbb{P}(A|B) & \text{if } \omega \in B \\ \mathbb{P}(A|B^c) & \text{if } \omega \in B^c \end{cases}$$

$$\{\omega \in \Omega : X(\omega) = \mathbb{P}(A|B)\} = B \in \sigma(\mathbb{1}_B)$$

$$\{X = \mathbb{P}(A|B^c)\} = B^c \in \sigma(\mathbb{1}_B)$$

$\therefore$ X is $\sigma(\mathbb{1}_B)$ measurable ✓

2 (Partial Averaging equation)

$$\sigma(\mathbb{1}_B) = \{\phi, \Omega, B, B^c\}$$

$$\mathbb{E}(X\mathbb{1}_B) = \mathbb{E}(\mathbb{P}(A|B)\mathbb{1}_B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}\mathbb{P}(B)$$

$$= \mathbb{P}(A\cap B) = \mathbb{E}(\mathbb{1}_A \mathbb{1}_B)$$

$$\mathbb{E}(X\mathbb{1}_{B^c}) \overset{\text{Same}}{=} \mathbb{E}(\mathbb{1}_A \mathbb{1}_{B^c})$$

$$\mathbb{E}(X\mathbb{1}_\phi) = 0 = \mathbb{E}(\mathbb{1}_A \mathbb{1}_\phi)$$

$$\mathbb{E}(X\mathbb{1}_\Omega) = \mathbb{E}(\mathbb{P}(A|B)\mathbb{1}_B + \mathbb{P}(A|B^c)\mathbb{1}_{B^c})$$

$$= \mathbb{P}(A\cap B) + \mathbb{P}(A\cap B^c) = \mathbb{P}(A)$$

$$= \mathbb{E}(\mathbb{1}_A \mathbb{1}_\Omega)$$

$$\therefore X = \mathbb{E}(\mathbb{1}_A | \sigma(\mathbb{1}_B))$$

Ex3: Martingale $\Longrightarrow$ constant expectation

$\not\Longleftarrow$

$$X_t = \frac{1+W_t^2}{1+t}, \quad W \text{ is a B.M.}$$

- $\mathbb{E}(X_t) = \dfrac{1+t}{1+t} = 1$

- $t \geq s$: $\underline{\mathbb{E}(X_t \mid \mathcal{F}_s)} = \mathbb{E}\left(\dfrac{1+W_t^2}{1+t} \,\Big|\, \mathcal{F}_s\right)$

$$\cancel{X_s} \quad = \mathbb{E}\left(\frac{1+(W_t-W_s+W_s)^2}{1+t} \,\Big|\, \mathcal{F}_s\right)$$

$$= \mathbb{E}\left(\frac{1+(W_t-W_s)^2+2W_s(W_t-W_s)+W_s^2}{1+t} \,\Big|\, \mathcal{F}_s\right)$$

$1+W_s^2 = X_s(1+s)$

$$= \frac{1}{1+t} + \frac{t-s}{1+t} + 0 + \frac{W_s^2}{1+t}$$

$$= \frac{W_s^2+1}{1+t} + \frac{t-s}{1+t} = \boxed{\frac{X_s(1+s)}{1+t} + \frac{t-s}{1+t}} \overset{?}{\cancel{=}} X_s$$

$$\Leftrightarrow \cancel{\frac{t-s}{1+t}} = \cancel{\frac{(t-s)X_s}{1+t}} \Leftrightarrow X_s = 1 \Leftrightarrow W_s^2 = s$$

Impossible!!!

$$X_s - \frac{X_s(1+s)}{1+t} = X_s\left(\frac{1+t}{1+t} - \frac{1+s}{1+t}\right)$$

Given $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, M martingale, $M_0 = 0$

Define

$$\Delta_t = \begin{cases} \Delta_0 & \text{if } t \leq t_0 \\ \Delta_{t_0} & \text{if } t > t_0 \end{cases}$$

given

Simple process

Δ_{t_0} is $\mathcal{F}_{t_0}$-mble

$$I_t = \Delta_0 (M_{t \wedge t_0} - \underset{}{\overset{0}{\overset{\parallel}{M_0}}}) + \Delta_{t_0} (M_t - M_{t \wedge t_0})$$

$a \wedge b = \min\{a, b\}$

M – stock price

Δ – trading strat

Show I_t is a martingale. (You check)

Ex1 Given $X, Y \sim N(0,1)$. $X \perp\!\!\!\perp Y$.

$$\mathbb{E}\left(e^{\frac{1}{2}XY}\right)$$

$$\mathcal{G} = \sigma(X)$$

$$\mathbb{E}\Big(\underbrace{\mathbb{E}\left(e^{\frac{1}{2}XY} \mid \mathcal{G}\right)}_{= g(X) = e^{\frac{X^2}{8}}}\Big) \qquad f(x,y) = e^{\frac{1}{2}xy}$$

By I.L.,

$$\mathbb{E}\left(e^{\frac{1}{2}XY} \mid \mathcal{G}\right) = g(X)$$

$$g(x) = \mathbb{E}\left(e^{\frac{1}{2}xY}\right) = e^{\frac{1}{2}\cdot\left(\frac{1}{2}x\right)^2} = e^{\frac{x^2}{8}}$$

3° Independence lemma

Given $X, Y, \mathcal{G}$.

① X is $\mathcal{G}$-measurable

② $Y \perp\!\!\!\perp \mathcal{G}$. ($\sigma(Y) \perp\!\!\!\perp \mathcal{G}$)

Let f continuous.

$$\mathbb{E}(f(X,Y) \mid \mathcal{G}) = g(X)$$

where $g(x) = \mathbb{E}(f(x, Y))$

(You check)

$$\mathbb{E}\left(e^{\frac{X^2}{8}}\right) = \int_{-\infty}^{\infty} e^{\frac{z^2}{8}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz = \frac{2}{\sqrt{3}}$$

<u>Ex2</u>: $X \sim \mathrm{Exp}(1)$, Y is $N(0,1)^2$, $X \perp\!\!\!\perp Y$.

$$Z = \frac{X}{Y}$$

$$\mathbb{P}(Z \geq t) = ?$$

$\mathcal{G} = \sigma(X)$, $\underline{\mathcal{G} = \sigma(Y)}$

$g(y) = \underline{e^{-ty}}$ $\quad \underline{\mathbb{E}(e^{-tY})}$

$$\mathbb{P}(Z \geq t) = \mathbb{P}(X \geq tY) = \mathbb{E}\left(\mathbb{1}_{\{X \geq tY\}}\right)$$

$$= \mathbb{E}\Big(\underbrace{\mathbb{E}\left(\mathbb{1}_{\{X \geq tY\}} \mid \mathcal{G}\right)}_{= g(Y)}\Big)$$

$$g(y) = \mathbb{E}\left(\mathbb{1}_{\{X \geq ty\}}\right) = \mathbb{P}(X \geq ty) = \underline{1 - \mathbb{P}(X < ty)}$$

$X \sim \mathrm{Exp}(1)$

$$= 1 - (1 - e^{-ty})$$

$$= e^{-ty}$$